Complex Variables

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Use Cauchy's integral formula to evaluate $\displaystyle\int_{|z-1|=1} \frac{\cos(2\pi z)}{z^2-1}\,dz$, where the integration path is oriented in the standard counterclockwise direction.

   Solution. Since $z^2 - 1 = (z-1)(z+1)$, the integral can be written as follows:

   $$\int_{|z-1|=1} \frac{(z+1)^{-1}\cos(2\pi z)}{z-1}\,dz = \int_{|z-1|=1} \frac{f(z)}{z-1}\,dz,$$

   where $f(z) = (z+1)^{-1}\cos(2\pi z)$. By Cauchy's integral formula, the integral equals $2\pi i f(1)$, which simplifies to πi.

2. Student Kan Toor says that if γ is any smooth curve starting at the point 1 and ending at the point i (and not passing through 0), then

   $$\int_\gamma \frac{1}{z}\,dz = \log(i) - \log(1) = \frac{i\pi}{2}.$$

   "I know there is an ambiguity in the value of the logarithm, but since I am taking a difference of two values, the ambiguity cancels out," says Kan Toor. Is Kan Toor's analysis correct? Explain why or why not.

   Solution. Kan Toor is wrong. Indeed, if the value of the integral were independent of the path, then the integral along a quarter circle in the first quadrant would equal the integral along three quarters of a circle in the remaining quadrants. It would follow that the integral of $1/z$ around a full closed circle would be 0, but we know that the integral of $1/z$ around the unit circle equals $2\pi i$.

   Kan Toor's answer is correct, however, for a path γ that remains in the first quadrant or in the right-hand half plane.

   In general, the integral equals i times the net change in the argument of z along the curve γ. The result could be $\frac{i\pi}{2} + 2\pi i n$ for any integer n (depending on the choice of the curve γ).

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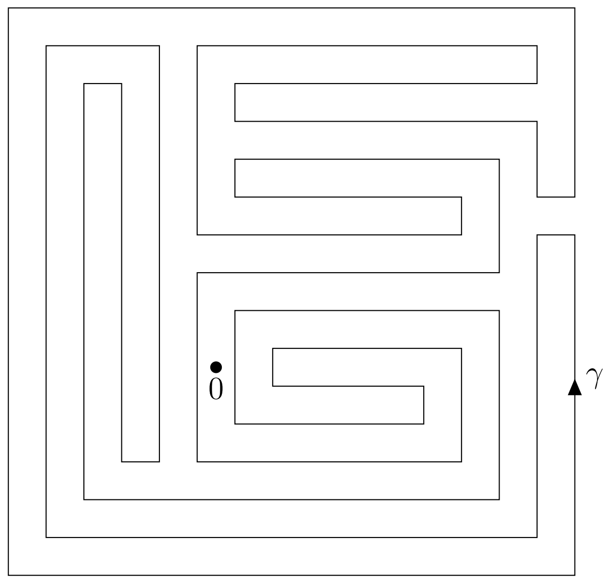
3. Student Ko Shi says that $\displaystyle\int_\gamma \frac{1}{z}\,dz = 2\pi i$, where γ is the simple closed curve indicated below.



Is Ko Shi right or wrong? Explain why.

Solution. Ko Shi would be right if the point 0 were inside the curve γ, but actually 0 is outside, so the integral is equal to 0 instead of $2\pi i$. You can see that the point 0 is outside either by shading the inside of the curve or by following the dashed line that connects 0 to the outside of the maze without crossing γ.