Complex Variables

Problem Show that $\displaystyle\int_0^{2\pi} \frac{(\sin(\theta))^2}{5-4\cos(\theta)}\,d\theta = \frac{\pi}{4}$.

Solution. The strategy is to "unparametrize" the integral by setting z equal to $e^{i\theta}$, which means that $d\theta = \frac{dz}{iz}$. Since $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, and $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{i\theta})$, the integral turns into the following path integral:

$$\int_{|z|=1} \frac{\left(\frac{1}{2i}(z-z^{-1})\right)^2}{5-2(z+z^{-1})}\,\frac{dz}{iz}, \qquad \text{or} \qquad \frac{1}{4i}\int_{|z|=1} \frac{z^4-2z^2+1}{z^2(2z^2-5z+2)}\,dz.$$

Since $2z^2 - 5z + 2 = (2z-1)(z-2)$, there are poles inside the unit circle at 0 and $1/2$.

To determine the residue at the simple pole at $1/2$, view the integrand as

$$\frac{(z^4-2z^2+1)/z^2}{2z^2-5z+2}$$

and apply the $g(z_0)/h'(z_0)$ rule. The residue is

$$\left.\frac{(z^4-2z^2+1)/z^2}{4z-5}\right|_{z=1/2}, \qquad \text{which simplifies to } -3/4.$$

The residue at the double pole at 0 equals the coefficient of z in the Maclaurin expansion of

$$\frac{z^4-2z^2+1}{2z^2-5z+2},$$

which is the same as the coefficient of z in the Maclaurin expansion of

$$\frac{1}{2z^2-5z+2}.$$

Using a geometric-series expansion shows that

$$\frac{1}{2z^2-5z+2} = \frac{1}{2}\cdot\frac{1}{1-(\frac{5}{2}z - z^2)} = \frac{1}{2}\cdot\left(1+\tfrac{5}{2}z+\cdots\right).$$

Consequently, the residue at 0 equals $5/4$.

The sum of the residues is $\frac{1}{2}$, so the residue theorem implies that the value of the original integral is $2\pi i \times \frac{1}{4i} \times \frac{1}{2}$, or $\frac{\pi}{4}$.

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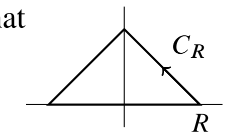
Problem Evaluate $\displaystyle\int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz$ and deduce that $\displaystyle\int_0^\infty \frac{\cos(x)}{(x^2+1)^2}\,dx = \frac{\pi}{2e}$.



Solution. When $R > 1$, the path encloses one singular point, a double pole at i. Since

$$\frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z-i)^2(z+i)^2},$$

the residue at this pole equals

$$\frac{d}{dz}\frac{e^{iz}}{(z+i)^2}\bigg|_{z=i}, \qquad \text{or} \qquad \frac{(z+i)^2 ie^{iz} - e^{iz}2(z+i)}{(z+i)^4}\bigg|_{z=i},$$

which simplifies to $-ie^{-1}/2$. By the residue theorem, the integral equals $2\pi i$ times this residue, or π/e.

On the other hand, the integral can be computed by parametrizing the path. The integral along the piece of the real axis equals

$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\,dx, \qquad \text{or} \qquad \int_{-R}^{R} \frac{\cos(x) + i\sin(x)}{(x^2+1)^2}\,dx,$$

which by symmetry considerations reduces to

$$2\int_0^R \frac{\cos(x)}{(x^2+1)^2}\,dx.$$

To handle the integral over the rest of the path, use the principle that the modulus of an integral is at most the length of the path times the maximum of the modulus of the integrand on the path. Now $|e^{iz}| = |e^{ix}e^{-y}| = e^{-y}$, and $e^{-y} \le 1$ when $y \ge 0$ (in particular, when z is on the part of C_R in the upper half-plane). And $|z| \ge R/\sqrt{2}$ on this part of the path, so $|z^2+1| \ge \frac{R^2}{2} - 1$ by the triangle inequality. Therefore

$$\left|\frac{e^{iz}}{(z^2+1)^2}\right| \le \frac{1}{(\frac{R^2}{2}-1)^2}$$

when $R > \sqrt{2}$, and z is on the part of C_R in the upper half-plane. The length of this part of the path is $2\sqrt{2}\,R$, so the integral over this part of the path has modulus not exceeding

$$\frac{2\sqrt{2}\,R}{(\frac{R^2}{2}-1)^2}, \qquad \text{which tends to 0 when } R \to \infty.$$

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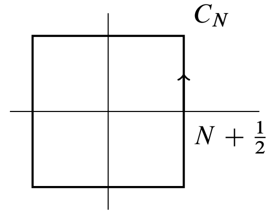
Consequently, taking the limit as $R \to \infty$ shows that

$$\frac{\pi}{e} = 2\int_0^\infty \frac{\cos(x)}{(x^2+1)^2}\,dx,$$

an equation that is equivalent to the statement of the problem.

Remark By the path-deformation principle, you could just as well integrate over a semicircle. An advantage of a triangle is that if you do parametrize the path explicitly, then you get a simpler expression in this problem with a triangle than with a semicircle. Indeed, with a semicircle, you would get a term $e^{iRe^{i\theta}}$ in the numerator, while with a triangle, you get a term e^{ix+x-R} when $0 < x < R$ (and a corresponding term e^{ix-x-R} when $-R < x < 0$).

Problem Evaluate $\displaystyle\int_{C_N} \frac{\pi}{z^2\sin(\pi z)}\,dz$ and deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.



Solution. The strategy is to apply the residue theorem. The integrand has a triple pole at 0 and simple poles at the nonzero integers. When n is a nonzero integer, view the integrand as $\frac{\pi/z^2}{\sin(\pi z)}$ and apply the $g(z_0)/h'(z_0)$ rule: the residue at n equals

$$\left.\frac{\pi/z^2}{\pi\cos(\pi z)}\right|_{z=n}, \qquad \text{or} \qquad \frac{1}{n^2\cos(\pi n)}, \qquad \text{or} \qquad \frac{(-1)^n}{n^2}.$$

Notice that the residues at n and at $-n$ are equal. One way to find the residue at the origin is to expand $\sin(\pi z)$ in a Maclaurin series:

$$\begin{aligned}\frac{\pi}{z^2\sin(\pi z)} &= \frac{\pi}{z^2(\pi z - \frac{1}{3!}(\pi z)^3 + \cdots)} = \frac{1}{z^3(1-\frac{\pi^2}{6}z^2+\cdots)} \\ &= \frac{1}{z^3}\left(1+\frac{\pi^2}{6}z^2+\cdots\right) \qquad \text{for } z \text{ close to } 0,\end{aligned}$$

where the final step is an application of the geometric-series expansion. Thus the residue, the coefficient of $1/z$ in the Laurent series, is equal to $\pi^2/6$. (A shout-out to WolframAlpha confirms the result via `Series[Pi/(z^2 Sin[Pi z]),{z,0}]` or directly via `Residue[Pi/(z^2 Sin[Pi z]),{z,0}]`.)

The residue theorem yields that

$$\int_{C_N} \frac{\pi}{z^2\sin(\pi z)}\,dz = 2\pi i\left(\frac{\pi^2}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2}\right).$$

The claim now is that the integral on the left-hand side tends to 0 when $N \to \infty$. It follows from this claim that

$$0 = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

and this equation is equivalent to the statement of the problem.

To verify the claim, use the principle that the modulus of an integral is at most the length of the path times the maximum of the modulus of the integrand on the path. The main point is that $|\sin(\pi z)|$ is bounded away from 0 on the path. On the right-hand edge of the path,

$$|\sin(\pi z)| = \left|\frac{1}{2i}\left(e^{\pi N i+\frac{\pi}{2}i-\pi y} - e^{-\pi N i-\frac{\pi}{2}i+\pi y}\right)\right| = \cosh(\pi y) \ge 1,$$

and similarly on the left-hand edge. The triangle inequality implies that on the top edge,

$$\begin{aligned}|\sin(\pi z)| &= \left|\frac{1}{2i}\left(e^{i\pi x}e^{-(N+\frac{1}{2})\pi} - e^{-i\pi x}e^{(N+\frac{1}{2})\pi}\right)\right| \\ &\ge \frac{1}{2}\left(e^{(N+\frac{1}{2})\pi} - e^{-(N+\frac{1}{2})\pi}\right) = \sinh\left[(N+\tfrac{1}{2})\pi\right].\end{aligned}$$

On the positive real axis, the hyperbolic sine function is increasing and tends to ∞, but for present purposes, it suffices to observe that $\sinh[(N+\frac{1}{2})\pi] \ge \sinh(\frac{3}{2}\pi)$ when $N \ge 1$, and $\sinh(\frac{3}{2}\pi) > 1$ (actually $\sinh(\frac{3}{2}\pi) > 55$). A parallel computation holds on the bottom edge. The upshot is that $|\sin(\pi z)| \ge 1$ everywhere on the path C_N, so $|1/\sin(\pi z)| \le 1$. Moreover $|z| > N$ on C_N, so $|1/z^2| < 1/N^2$. Thus

$$\left|\frac{\pi}{z^2\sin(\pi z)}\right| < \frac{\pi}{N^2} \qquad \text{on } C_N,$$

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and the length of the path C_N equals $4 \times 2 \times (N + \frac{1}{2})$, or $8N + 4$, so

$$\left| \int_{C_N} \frac{\pi}{z^2 \sin(\pi z)} \, dz \right| < \frac{(8N+4)\pi}{N^2}.$$

Consequently, the integral over C_N does tend to 0 when $N \to \infty$. This validation of the claim finishes the solution of the problem.

Remark Some students wondered if the series in the problem could be summed in a different way by using the following formula from the beginning of Section 7.10 in the textbook:

$$\csc(z) = \frac{1}{z} - 2z \left(\frac{1}{z^2 - \pi^2} - \frac{1}{z^2 - 4\pi^2} + \frac{1}{z^2 - 9\pi^2} - \cdots \right).$$

The derivation of this formula (not provided in the textbook) requires an argument similar to the preceding solution, so invoking the formula to solve the problem is a somewhat circular argument. But if you are willing to accept this formula, then the problem can be solved without any integration. Namely, rearrange the formula algebraically as follows:

$$\left(\frac{1}{\sin(z)} - \frac{1}{z} \right) \cdot \frac{1}{2z} = \frac{1}{\pi^2 - z^2} - \frac{1}{4\pi^2 - z^2} + \frac{1}{9\pi^2 - z^2} - \cdots.$$

When $z = 0$, the right-hand side becomes

$$\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2},$$

so the problem reduces to showing that the limit as $z \to 0$ of the left-hand side is equal to $1/12$. That calculation is tedious to carry out via l'Hôpital's rule, but easy using series:

$$\frac{1}{\sin(z)} = \frac{1}{z - \frac{1}{3!}z^3 + \cdots} = \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{6}z^2 + \cdots} = \frac{1}{z} \cdot \left(1 + \tfrac{1}{6}z^2 + \cdots\right),$$

so

$$\left(\frac{1}{\sin(z)} - \frac{1}{z} \right) \cdot \frac{1}{2z} = \frac{1}{12} + \text{ positive powers of } z$$

for z in a small punctured neighborhood of 0.