Computation of an Integral

On the first day of class, I mentioned the following problem as one that we would be able to solve by the end of the semester:

$$\int_0^\infty \frac{1}{x(1+x^2)} \log\left|\frac{x+\sqrt{3}}{x-\sqrt{3}}\right| \, dx = \frac{\pi^2}{6}. \tag{1}$$

My mother used this integral in her PhD dissertation on theoretical physics.

On the last day of class, I tried to work out the integral via residues and got the wrong answer by failing to pay careful attention to the branch of the logarithm. Here is a correct calculation.

Consider the fractional linear transformation $\dfrac{\sqrt{3}+z}{\sqrt{3}-z}$. Since the coefficients are real numbers, this transformation maps the (extended) real axis back to itself. Accordingly, the transformation must map the upper half-plane either to the upper half-plane or to the lower half-plane.

To determine which image is correct, locating the image of one point will suffice. A convenient point to test is the point i. Observe that

$$\frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{\sqrt{3}+i}{\sqrt{3}-i} \cdot \frac{\sqrt{3}+i}{\sqrt{3}+i} = \frac{3+2i\sqrt{3}-1}{3+1} = \frac{1+i\sqrt{3}}{2}. \tag{2}$$

Since the image of the point i lies in the upper half-plane, the linear fractional transformation maps the upper half-plane to the upper half-plane, not to the lower half-plane.

Accordingly, there is an analytic branch of $\log \dfrac{\sqrt{3}+z}{\sqrt{3}-z}$ defined when z lies in the upper half-plane, and the argument (angle) can be taken to lie between 0 and π. Moreover, this logarithm extends to be analytic at all points of the real axis except the points $\pm\sqrt{3}$.

Next consider the auxiliary problem of computing

$$\int_{C_R} \frac{1}{z(1+z^2)} \log\left(\frac{\sqrt{3}+z}{\sqrt{3}-z}\right) dz$$

around the closed curve C_R shown in the diagram on the next page. The cut-outs around the points $\pm\sqrt{3}$ are there to avoid the points where the logarithm is not analytic. There is no need for a cut-out around 0, for the apparent singularity at 0 is removable: the fractional linear transformation maps 0 to 1, and $\log(1) = 0$.

By the residue theorem, the value of the integral around C_R equals

$$2\pi i \operatorname{Res}\left(\frac{\frac{1}{z}\log\left(\frac{\sqrt{3}+z}{\sqrt{3}-z}\right)}{1+z^2}, i\right) = 2\pi i \frac{\frac{1}{z}\log\left(\frac{\sqrt{3}+z}{\sqrt{3}-z}\right)}{2z}\Bigg|_{z=i} = \frac{\pi}{i}\log\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right).$$

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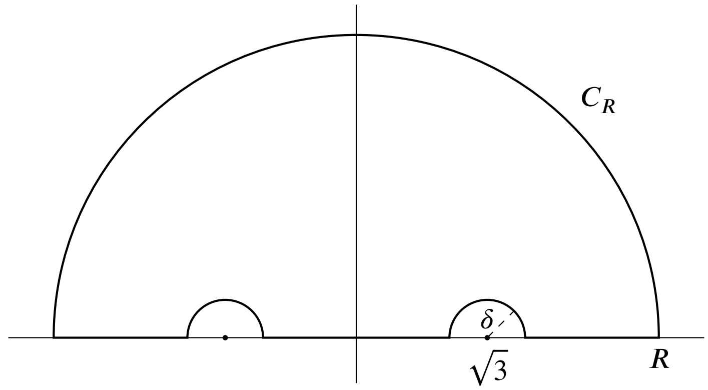

Now equation (2) implies that

$$\log\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right) = \log\left|\frac{\sqrt{3}+i}{\sqrt{3}-i}\right| + i\arg\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right) = \log(1) + i\arg\left(\frac{1+i\sqrt{3}}{2}\right) = \frac{i\pi}{3}.$$

Therefore the integral over C_R equals $\frac{\pi}{i}\cdot\frac{i\pi}{3}$, or $\frac{\pi^2}{3}$.

Showing that the integrals over the three semicircles tend to 0 when $R\to\infty$ and $\delta\to 0$ will justify the conclusion that

$$\frac{\pi^2}{3} = \int_{-\infty}^{\infty} \frac{1}{x(1+x^2)} \log\left(\frac{\sqrt{3}+x}{\sqrt{3}-x}\right) dx.$$

Taking the real part of both sides (which does not change the left-hand side) reveals that

$$\frac{\pi^2}{3} = \int_{-\infty}^{\infty} \frac{1}{x(1+x^2)} \log\left|\frac{x+\sqrt{3}}{x-\sqrt{3}}\right| dx. \tag{3}$$

The integrand in (3) surprisingly is an even function, because

$$\log\left|\frac{-x+\sqrt{3}}{-x-\sqrt{3}}\right| = \log\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| = -\log\left|\frac{x+\sqrt{3}}{x-\sqrt{3}}\right|,$$

so the integrand has two compensating odd factors. Invoking this symmetry and dividing equation (3) by a factor of 2 produces the required equation (1).

What remains in the calculation is to bound the integrals over the semicircles. First consider the integral over the large semicircle of radius R. The length of the path is πR, so the problem reduces to bounding the integrand by some expression that tends to 0 faster than $1/R$. Evidently

$$\left|\frac{1}{z(1+z^2)}\right| \le \frac{1}{R(R^2-1)} \qquad \text{when } R \text{ is large,}$$

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which looks hopeful, but how big is the absolute value of the logarithm term? By the definition of the logarithm and by the triangle inequality,

$$\left|\log\left(\frac{\sqrt{3}+z}{\sqrt{3}-z}\right)\right| = \left|\log\left|\frac{\sqrt{3}+z}{\sqrt{3}-z}\right| + i\arg\left(\frac{\sqrt{3}+z}{\sqrt{3}-z}\right)\right| \le \left|\log\left|\frac{\sqrt{3}+z}{\sqrt{3}-z}\right|\right| + \pi, \tag{4}$$

since the fractional linear transformation takes values in the upper half-plane, where the angle is between 0 and π. Notice that the double absolute value signs are needed, because $\log|w|$ might be a negative real number. When $z \to \infty$, the value of the fractional linear transformation approaches -1, so the logarithm of the absolute value approaches 0. Consequently, the expression (4) is bounded above by any number larger than π, say by 4, when R is large enough. The upshot is that the integral over the big semicircle can be bounded when R is large by

$$\frac{4\pi R}{R(R^2-1)},$$

and this bound certainly tends to 0 when $R \to \infty$.

Next consider the integral over the small semicircle of length $\pi\delta$ centered at $\sqrt{3}$. Since

$$\lim_{z\to\sqrt{3}} \frac{1}{z(1+z^2)} = \frac{1}{4\sqrt{3}},$$

the function $\dfrac{1}{z(1+z^2)}$ certainly has absolute value less than 1 when z is close to $\sqrt{3}$, that is, when the radius δ of the small semicircle is close to 0. What about the logarithm? Parametrizing the small semicircle by setting z equal to $\sqrt{3}+\delta e^{i\theta}$ with θ between 0 and π shows that

$$\left|\log\left(\frac{\sqrt{3}+z}{\sqrt{3}-z}\right)\right| = \left|\log\left(\frac{2\sqrt{3}+\delta e^{i\theta}}{-\delta e^{i\theta}}\right)\right| < \pi + \log\frac{4}{\delta}$$

when δ is so small that $2\sqrt{3}+\delta < 4$. The conclusion is that if δ is close enough to 0, then the integral over the small semicircle is bounded by

$$\pi\delta\cdot 1\cdot\left(\pi+\log\frac{4}{\delta}\right).$$

The limit of this expression when $\delta \to 0$ equals 0 because $\lim_{\delta\to 0^+}\delta\log(1/\delta) = 0$, as you can check by applying l'Hôpital's rule to the equivalent fraction

$$\frac{\log\left(\frac{1}{\delta}\right)}{\frac{1}{\delta}}.$$

The small semicircle centered at $-\sqrt{3}$ can be handled in the same way. Since the integrals over all three semicircles have limit 0, the calculation is complete.