Complex Variables

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. State the following:
   a) Euler's formula (relating the exponential and trigonometric functions); and
   b) the power series expansion for $\sin(z)$ (centered at 0).

Solution.

a) Euler's formula says that $e^{i\theta} = \cos(\theta) + i\,\sin(\theta)$.

b) The Maclaurin series for $\sin(z)$ is

$$z - \tfrac{1}{3!}z^3 + \tfrac{1}{5!}z^5 + \cdots \qquad \text{or} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

2. Determine the smallest positive integer n such that

$$\left(\sqrt{3} + i\right)^n = \left(1 + i\sqrt{3}\right)^n.$$

Solution. Since $\sqrt{3} + i = 2e^{i\pi/6}$ and $1 + i\sqrt{3} = 2e^{i\pi/3}$, the question reduces to finding n for which $e^{i\pi n/6} = e^{i\pi n/3}$, or $1 = e^{i\pi n/3 - i\pi n/6} = e^{i\pi n/6}$. The smallest such positive n has the property that $\pi n/6 = 2\pi$, or $n = 12$.

3. The set of all complex numbers z such that

$$\operatorname{Re}\left(\frac{1-z}{1+z}\right) = 1$$

can be represented in the plane as a certain curve. What curve is it?
Caution: The real part of a quotient is *not* equal to the quotient of the real parts!

Solution. **Method 1** To find the real part of the quotient, start by multiplying both the numerator and the denominator by the complex conjugate of the denominator:

$$\frac{1-z}{1+z} = \frac{1-x-iy}{1+x+iy} \cdot \frac{1+x-iy}{1+x-iy} = \frac{1-x^2-y^2-2iy}{(1+x)^2+y^2}.$$

Therefore $\operatorname{Re}\left(\frac{1-z}{1+z}\right) = \frac{1-x^2-y^2}{(1+x)^2+y^2}$. Setting this expression equal to 1 and clearing the denominator shows that

$$1-x^2-y^2 = (1+x)^2+y^2.$$

Simplifying yields that $0 = x + x^2 + y^2$, and completing the square shows that $\frac{1}{4} = (x+\frac{1}{2})^2 + y^2$. This equation represents a circle of radius $\frac{1}{2}$ centered at the point $(-\frac{1}{2}, 0)$. But the point $(-1, 0)$ is missing from the circle, because the original equation is undefined when $z = -1$.

Method 2 A bit of algebraic trickery reduces the amount of calculation required. Since

$$\frac{1-z}{1+z} = \frac{1+z-2z}{1+z} = 1 - \frac{2z}{1+z},$$

the real part of $\frac{1-z}{1+z}$ equals 1 precisely when the real part of $\frac{z}{1+z}$ equals 0. In other words, the quantity $\frac{z}{1+z}$ is purely imaginary. An equivalent property, as long as $z \neq -1$ and $z \neq 0$, is that the reciprocal $\frac{1+z}{z}$ is purely imaginary, or $\operatorname{Re}(1/z) = -1$. Now

$$\frac{1}{z} = \frac{x-iy}{x^2+y^2}, \qquad \text{so} \qquad \operatorname{Re}\frac{1}{z} = \frac{x}{x^2+y^2},$$

and setting this expression equal to -1 leads again to the condition that $0 = x + x^2 + y^2$, as obtained in Method 1. The case that $z = -1$ has to be excluded, as before. On the other hand, the special case that $z = 0$ evidently does satisfy the original equation.

4. Let $f(z)$ denote an analytic function with real part $u(x, y)$ and imaginary part $v(x, y)$. Determine $f(z)$ if

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 6xy + 1 \qquad \text{and} \qquad f(0) = 0.$$

Solution. **Method 1** More information is given than is needed to solve the problem. Taking advantage of this extra information makes it possible to give a short solution.

Since $f' = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$, the given information implies that

$$f'(x+iy) = 3x^2 - 3y^2 + i(6xy+1).$$

Then $f'(x+i0) = 3x^2 + i$, so $f'(z) = 3z^2 + i$, and integrating shows that $f(z) = z^3 + iz + C$ for some constant C. But $f(0) = 0$, so $C = 0$. Therefore $f(z) = z^3 + iz$.

Method 2 Since the derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ are polynomials in x and y of degree 2, the function $f(z)$ must be a polynomial in z of degree 3. Moreover, since $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ have no first-degree terms, the function $f(z)$ must be missing its quadratic term. And the constant term in $f(z)$ is missing too, since $f(0) = 0$. Accordingly,

$$\begin{aligned}
f(x+iy) &= (a_3+ib_3)(x+iy)^3 + (a_1+ib_1)(x+iy)\\
&= a_3x^3 - 3b_3x^2y - 3a_3xy^2 + b_3y^3 + a_1x - b_1y\\
&\quad + i(b_3x^3 + 3a_3x^2y - 3b_3xy^2 - a_3y^3 + a_1y + b_1x),
\end{aligned}$$

where a_3, b_3, a_1, and b_1 are real constants to be determined. Then

$$\frac{\partial u}{\partial x} = 3a_3x^2 - 6b_3xy - 3a_3y^2 + a_1,$$

so comparing with the given expression for $\dfrac{\partial u}{\partial x}$ shows that $a_3 = 1$, $b_3 = 0$, and $a_1 = 0$. Similarly,

$$\frac{\partial v}{\partial x} = 3b_3x^2 + 6a_3xy - 3b_3y^2 + b_1,$$

and comparing with the given expression for $\dfrac{\partial v}{\partial x}$ confirms that $b_3 = 0$ and $a_3 = 1$ and shows that $b_1 = 1$. Therefore $f(z) = z^3 + iz$.

Method 3 Integrating the first equation with respect to x shows that $u(x,y) = x^3 - 3xy^2 + C(y)$, where $C(y)$ is a function of y to be determined. Then

$$\frac{\partial u}{\partial y} = -6xy + C'(y).$$

But $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ by the Cauchy–Riemann equations, so comparing with the second given equation shows that $C'(y) = -1$, or $C(y) = -y + b$ for some constant b to be determined. In other words, $u(x,y) = x^3 - 3xy^2 - y + b$. The given information that $f(0) = 0$ implies that $u(0,0) = 0 = v(0,0)$, so $b = 0$ and $u(x,y) = x^3 - 3xy^2 - y$.

Similarly, since $\dfrac{\partial v}{\partial x} = 6xy + 1$, integrating with respect to x shows that $v(x, y) = 3x^2y + x + A(y)$ for a function $A(y)$ to be determined. Then

$$\frac{\partial v}{\partial y} = 3x^2 + A'(y).$$

But $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$ by the Cauchy–Riemann equations, so comparing with the first given equation shows that $A'(y) = -3y^2$, or $A(y) = -y^3$ (the integration constant is 0 as before, since $v(0, 0) = 0$). Thus $v(x, y) = 3x^2y + x - y^3$.

Putting the above deductions together shows that $f(x + iy) = u(x, y) + iv(x, y) = x^3 - 3xy^2 - y + i(3x^2y + x - y^3)$. In particular, $f(x + i0) = x^3 + ix$. Since $f(z)$ evidently is a polynomial in z, it follows that $f(z) = z^3 + iz$.

5. Evaluate the limit

$$\lim_{z\to 0} (\cos z)^{1/z^2}$$

(using the principal branch of the logarithm).

Solution. By definition, $(\cos z)^{1/z^2} = \exp\left(\frac{1}{z^2}\log\cos z\right)$, and the exponential function is continuous, so what needs to be computed is

$$\exp\left(\lim_{z\to 0} \frac{\log\cos z}{z^2}\right).$$

Method 1 Since $\cos(0) = 1$, and the principal branch of $\log(1)$ equals 0, l'Hôpital's rule can be invoked to say that

$$\lim_{z\to 0} \frac{\log\cos z}{z^2} = \lim_{z\to 0} \frac{-\sin(z)/\cos(z)}{2z}.$$

Now either apply l'Hôpital's rule a second time or use the known result that $\displaystyle\lim_{z\to 0} \frac{\sin(z)}{z} = 1$ to conclude that

$$\lim_{z\to 0} \frac{-\sin(z)/\cos(z)}{2z} = -\frac{1}{2}.$$

Accordingly, the answer to the original problem is $e^{-1/2}$.

Method 2 Use series expansions to say that

$$\frac{\log\cos z}{z^2} = \frac{\log(1 - \frac{1}{2}z^2 + O(z^4))}{z^2} = \frac{-\frac{1}{2}z^2 + O(z^4)}{z^2} = -\tfrac{1}{2} + O(z^2),$$

where the symbol $O(z^k)$ stands for terms of order z^k or higher. Therefore

$$\lim_{z\to 0} \frac{\log \cos z}{z^2} = -\frac{1}{2},$$

and the answer to the original problem is $e^{-1/2}$.

6. The hyperbolic tangent function, tanh, can be defined as follows:

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

For which values of the complex variable z is $\tanh(z)$ *not* analytic? In other words, what are the singular points of the hyperbolic tangent function?

Solution. There are singular points where the denominator is equal to 0. Now $e^z + e^{-z} = 0$ if and only if $e^z = -e^{-z}$, or $e^{2z} = -1$. Since $-1 = e^{\pi i + 2n\pi i}$, it follows that

$$2z = \pi i + 2n\pi i$$

for an arbitrary integer n, so $z = \frac{1}{2}\pi i + n\pi i$ for an arbitrary integer n.

Extra credit

Creatures from the galaxy Mocplex say that a function $u(x, y)$ is *morhanic* if

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

Are there any nonconstant analytic functions that have morhanic real part?

Solution. Yes. If $f(z) = (1 + i)az + c + id$ (where a, c, and d are real constants), then f is analytic, and the real part $u(x, y)$ equals $ax - ay + c$; evidently

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = a - a = 0,$$

so u is morhanic.

Complex Variables

The indicated form for f is the most general possible. To see why, observe that if u is morhanic, then differentiating the defining property first with respect to x and then with respect to y shows that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Consequently,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y}.$$

But also $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, since the real part of an analytic function is harmonic, so the preceding equation implies that all second-order partial derivatives of u are identically equal to 0. Therefore $u(x, y)$ must be a polynomial in x and y of degree at most 1: namely, $u(x, y) = ax + by + c$ for certain real constants a, b, and c. Since u is morhanic, b must equal $-a$. Now $x = \operatorname{Re}(z)$ and $-y = \operatorname{Re}(iz)$, so $u(x, y) = \operatorname{Re}(az + aiz + c)$. Therefore $f(z) = (1 + i)az + c$ plus an arbitrary imaginary constant.