Quiz

1. Find real numbers a and b satisfying the property that $(a+bi)^2 = i$.

   Solution. Expanding the left-hand side shows that $a^2 + 2abi - b^2 = i$. Therefore

   $$a^2 - b^2 = 0, \qquad \text{and}$$
   $$2ab = 1.$$

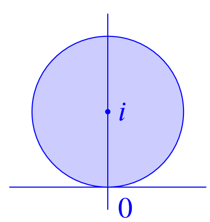
   The first equation implies that $a = \pm b$. If a were equal to $-b$, then the second equation would imply that $-2b^2 = 1$, which is impossible, since $-2b^2 \le 0$. Therefore a must be equal to b.

   The second equation now implies that $2b^2 = 1$, so $b = \pm 1/\sqrt{2}$. Thus the problem has two solutions: either $a = b = 1/\sqrt{2}$ or $a = b = -1/\sqrt{2}$.

   An alternative method is to express the complex number i as $e^{i\pi/2}$. Since $a + bi$ represents $\sqrt{i}$, or $i^{1/2}$, one solution for $a + bi$ is $e^{i\pi/4}$. This complex number can be rewritten as $\cos(\pi/4) + i\sin(\pi/4)$, or as $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$. (The other solution for $a + bi$ is the negative of this answer.)

2. Sketch a picture representing the set of values of the complex variable z for which $|z - i| \le 1$.

   Solution. The absolute value of a difference represents the distance between two complex numbers, so the distance between z and i is no greater than 1. In other words, the allowed values of z fill up a disk of radius 1 centered at i, as illustrated in the diagram. The boundary circle is included in the set.



3. If z is an element of the complex numbers satisfying the property that $|\operatorname{Re}(z)| = |z|$, then what can you deduce about z?

   Solution. The equation implies that the complex number z must actually be a real number. Indeed, if $\operatorname{Im} z$ were not 0, then $(\operatorname{Im} z)^2$ would be strictly positive, whence

   $$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} > \sqrt{(\operatorname{Re} z)^2} = |\operatorname{Re} z|,$$

   and the hypothesis that $|\operatorname{Re} z| = |z|$ would be violated. Thus $\operatorname{Im} z$ must be equal to 0.

   The geometric picture is that $|z|$ represents the distance from z to 0, and $|\operatorname{Re} z|$ represents the distance from z to the imaginary axis. By the triangle inequality, these two distances cannot be equal unless z lies on the real axis.