

Sample homework solutions

Most homework problems admit several methods of solution. These sample proofs illustrate different techniques: reducing to a previously solved problem, adapting a previously used method, enumerating cases, induction.

Problem 1 (number 9, page 5, §1.1). *Let x_1 , x_2 , and x_3 be real numbers. Prove that*

$$|x_1 + x_2 + x_3| \leq |x_1| + |x_2| + |x_3|. \quad (1)$$

Here are two different methods of proof.

Solution by applying a previous proposition. Using the associative law for addition, we may group $(x_1 + x_2 + x_3)$ as $(x_1 + x_2) + x_3$. The triangle inequality (Proposition 1.1.2(c)) implies that

$$|(x_1 + x_2) + x_3| \leq |x_1 + x_2| + |x_3|. \quad (2)$$

A second application of the triangle inequality yields that

$$|x_1 + x_2| \leq |x_1| + |x_2|. \quad (3)$$

By axiom (O4), we may add $|x_3|$ to both sides of (3) to deduce that

$$|x_1 + x_2| + |x_3| \leq |x_1| + |x_2| + |x_3|. \quad (4)$$

Applying the transitive law for inequality (axiom (O3)) to (2) and (4) yields the required result (1). \square

Solution from first principles. It is convenient to begin by proving an auxiliary result.

Lemma. *If x , y , z , and w are real numbers such that $x \leq y$ and $z \leq w$, then $x + z \leq y + w$.*

Proof. If $x \leq y$, then axiom (O4) implies that $x + z \leq y + z$. Moreover, if $z \leq w$, then axiom (O4) implies that $y + z \leq y + w$. The transitive law (O3) for inequality then implies the desired conclusion that $x + z \leq y + w$. \square

Now we proceed with the main argument. For every real number x (either positive, negative, or zero), the definition of absolute value implies that $x \leq |x|$. Consequently, $x_1 \leq |x_1|$ and $x_2 \leq |x_2|$ and $x_3 \leq |x_3|$. Applying the lemma twice, we deduce from these three inequalities that

$$x_1 + x_2 + x_3 \leq |x_1| + |x_2| + |x_3|. \quad (5)$$

This inequality (5) holds for all real numbers x_1, x_2 , and x_3 . In particular, the inequality still holds when we replace all three numbers by their negatives: namely,

$$-x_1 - x_2 - x_3 \leq |-x_1| + |-x_2| + |-x_3|,$$

or equivalently,

$$-(x_1 + x_2 + x_3) \leq |x_1| + |x_2| + |x_3|. \quad (6)$$

Combining inequalities (5) and (6) yields the desired inequality (1). \square

Problem 2 (number 3(c), page 19, §1.3). *Prove that the union of two countable sets is countable.*

Solution. Suppose S and T are infinite sets that are countable. This means that there exists a bijective function (a one-to-one correspondence) $f : S \rightarrow \mathbb{N}$, and there exists a bijective function $g : T \rightarrow \mathbb{N}$. To show that the union $S \cup T$ is countable, we need to show that there exists a bijective function $h : S \cup T \rightarrow \mathbb{N}$.

If the sets S and T are *disjoint* sets, then we can exhibit such a bijective function h as follows:

$$h(x) := \begin{cases} 2 \cdot f(x) & \text{if } x \in S, \\ 2 \cdot g(x) - 1 & \text{if } x \in T. \end{cases}$$

This function maps the elements of S one-to-one onto the even natural numbers and the elements of T one-to-one onto the odd natural numbers. Therefore h is a one-to-one correspondence from the union $S \cup T$ to the natural numbers. Thus we have proved that the union of two disjoint countable sets is countable.

If S and T are not disjoint, then we need to modify the preceding idea. Here are two possible ways to finish the proof.

Method 1 Define a function $\varphi : S \cup T \rightarrow \mathbb{N}$ as follows:

$$\varphi(x) := \begin{cases} 2 \cdot f(x) & \text{if } x \in S, \\ 2 \cdot g(x) - 1 & \text{if } x \in T \setminus S. \end{cases}$$

This function φ is one-to-one for the same reason that h is one-to-one, but φ is not necessarily surjective (onto). In other words, φ is a one-to-one correspondence between $S \cup T$ and a *subset* of the natural numbers. The range of φ , being a subset of \mathbb{N} , is countable by Proposition 1.3.2. Consequently, there exists a bijection θ from the range of φ onto \mathbb{N} . The composite map $\theta \circ \varphi$ is the required one-to-one correspondence between $S \cup T$ and the natural numbers \mathbb{N} .

Method 2 We can reduce the problem to the previously solved case of disjoint countable sets. Denote the set difference $T \setminus S$ by U . Observe that S and U are disjoint sets, and $S \cup T = S \cup U$. If U is an *infinite* subset of the countable set T , then U is countable by Proposition 1.3.2. In that case, the union $S \cup U$ of disjoint sets is countable by the argument above, and so $S \cup T$ is countable.

It remains to consider the case that U is a *finite* set. If U is the empty set, then S and $S \cup T$ are the same set, so $S \cup T$ is countable since S is. If U is a non-empty finite set with, say, n elements, then there is a bijective function $\rho : U \rightarrow \{1, \dots, n\}$ that enumerates the elements of U . We can define a bijection from $S \cup T$ to \mathbb{N} in terms of ρ and a bijection $f : S \rightarrow \mathbb{N}$ as follows:

$$x \mapsto \begin{cases} \rho(x) & \text{if } x \in U, \\ f(x) + n & \text{if } x \in S. \end{cases}$$

Thus we have shown in all cases that there exists a one-to-one correspondence between $S \cup T$ and the natural numbers. \square

Problem 3 (number 6, page 24, §1.4). *Prove that if n is a positive integer, then $n^3 + 5n$ is divisible by 6.*

Here are three different ways to execute the proof.

Solution by induction. Since $1^3 + 5 \cdot 1 = 6$, and 6 is obviously divisible by 6, the basis step ($n = 1$) of the induction holds. Now suppose that for a certain

positive integer n , it is known that $n^3 + 5n$ is divisible by 6. We need to deduce that $(n + 1)^3 + 5(n + 1)$ is divisible by 6. Expanding this quantity, collecting terms, and factoring, we find that

$$(n + 1)^3 + 5(n + 1) = (n^3 + 5n) + 3n(n + 1) + 6. \quad (7)$$

By the induction hypothesis, $(n^3 + 5n)$ is divisible by 6. On the other hand, $3n(n + 1)$ is obviously divisible by 3, and since one of the consecutive integers n and $n + 1$ is even, $3n(n + 1)$ is also divisible by 2. Thus all three terms on the right-hand side of (7) are divisible by 6, and so the left-hand side is divisible by 6. This completes the induction step of the proof. \square

Solution by clever algebra. Add and subtract n and factor to see that

$$n^3 + 5n = (n^3 - n) + 6n = n(n^2 - 1) + 6n = (n - 1)n(n + 1) + 6n. \quad (8)$$

Of three consecutive integers, exactly one is divisible by 3, and at least one is divisible by 2, so the product of three consecutive integers is always divisible by 6. Consequently, the right-hand side of (8) is divisible by 6, and so the left-hand side is too. \square

Solution by enumeration of cases. To show that $n^3 + 5n$ is divisible by 6, it suffices to show that this quantity is divisible both by 2 and by 3.

To show that $n^3 + 5n$ is divisible by 2, we need check only two cases: either n is even or n is odd. If n is even, then so are both n^3 and $5n$, so the sum $n^3 + 5n$ is even. If n is odd, then so are both n^3 and $5n$, so the sum $n^3 + 5n$ is even. Thus in both cases, $n^3 + 5n$ is divisible by 2.

To show that $n^3 + 5n$ is divisible by 3, we can check three cases: n is of the form $3k$ or $3k + 1$ or $3k + 2$ for some integer k . In the first case, $n^3 + 5n = 27k^3 + 15k$, which is divisible by 3. In the second case, $n^3 + 5n = 27k^3 + 27k^2 + 24k + 6$, which is divisible by 3. In the third case, $n^3 + 5n = 27k^3 + 54k^2 + 51k + 18$, which is divisible by 3. Thus in all cases, $n^3 + 5n$ is divisible by 3. \square

Remark. The theorem holds even without the hypothesis that n is positive. Indeed, if $n = -m$, then $n^3 + 5n = -(m^3 + 5m)$, so the result for negative n follows directly from the case of positive n .