Math 409

Advanced Calculus I

Fall 2004

Final Examination

1. State the following three theorems.
   (a) the Bolzano-Weierstrass theorem (about sequences)
   (b) the intermediate value theorem (about continuous functions)
   (c) the mean-value theorem (about differentiable functions)

2. Define the following three notions.
   (a) compact interval
   (b) Cauchy sequence
   (c) Riemann sum

3. (a) State the definition of "the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit L" in the form "for every $\epsilon > 0 \ldots$".
   (b) Prove from the definition that the sequence $\left\{\dfrac{1}{2^n}\right\}_{n=1}^{\infty}$ converges to the limit 0.

4. (a) State the definition of "the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges to the sum S".
   (b) Prove from the definition that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to the sum 1.

5. (a) State the definition of "the function f is continuous at the point x_0" in the form "for every $\epsilon > 0 \ldots$".
   (b) Prove from the definition that the function $f(x) = x^2$ is continuous at every point x_0.

6. (a) State the definition of "the function f is differentiable at the point a".

   (b) Prove from the definition that the function $f(x) = x^2$ is differentiable at every point a.

7. (a) State the definition of "the function f is integrable (in the sense of Riemann) on the interval $[a, b]$" in terms of upper sums and lower sums.

   (b) Prove from the definition that the function $f(x) = x^2$ is integrable on the interval $[0, 1]$.

8. This question concerns the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. In answering this question, you may cite relevant theorems from the course.

   (a) Show that the series converges for every x in the closed interval $[-1, 1]$.

   (b) Does the series represent an integrable function on the closed interval $[-1, 1]$? Explain why or why not.