Math 409-502

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The mean-value theorem revisited

Theorem. If f is a continuous function on a compact interval $[a, b]$, and if f is differentiable at all interior points of the interval, then there exists an interior point c for which $f(b) = f(a) + f'(c)(b-a)$.

Quadratic generalization. Suppose the derivative f' is continuous on $[a, b]$ and the second derivative f'' exists on (a, b). Then there exists an interior point c for which
$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(c)(b-a)^2$.

Example: Robinson Crusoe's approximation for $\ln(1.1)$**.**
Take $f(x) = \ln(x)$, $b = 1.1$, $a = 1$. Then there is a number c between 1 and 1.1 for which $\ln(1.1) = \ln(1) + \frac{1}{1}(0.1) + \frac{1}{2}(-\frac{1}{c^2})(0.1)^2$. Therefore $\ln(1.1) \approx 0.10$, and the exact value is smaller by an amount less than 0.005.
(The exact value of $\ln(1.1)$ is about 0.09531.)

Proof of quadratic mean-value theorem

Let g be the difference between f and the parabola that intersects the graph of f at a and b and has the same slope as f at a:

$$g(x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)^2 \right).$$

Then $g(a) = 0$ and $g(b) = 0$. By the original mean-value theorem, there exists a point c_1 between a and b for which $g'(c_1) = 0$. But $g'(a) = 0$, so there exists a point c_2 between a and c_1 for which $g''(c_2) = 0$. Then
$f''(c_2) = \frac{f(b)-f(a)-f'(a)(b-a)}{(b-a)^2} \cdot 2$,
which simplifies to the required equation.

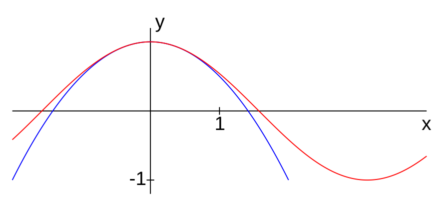
Higher-order mean-value theorem

Taylor's theorem. Suppose the function f has at least $(n+1)$ derivatives on an interval. If x and a are points of the interval, then there is some point c between x and a for which
$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$.

Homework

1. Read sections 17.1–17.3, pages 231–236.

2. Suppose you were to plot the functions $y = \cos(x)$ and $y = 1 - \frac{x^2}{2}$ on the same graph with the x and y axes scaled in inches (1 inch = 1 radian) using a line thickness of 1 point (where 1 inch = 72 points).



Over what interval of the x-axis would the two curves be indistinguishable? Why?