Math 409-502

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Announcement

Math Club Meeting today

Monday, November 15 at 7:30 PM

Blocker 156

Speaker: Jeff Nash, Aggie former student, from Veritas DGC

Title: Math and Seismic Imaging

FREE FOOD

Taylor's theorem

If f is $(n+1)$ times differentiable on an interval, and if x and a are points of the interval, then there is some point c between x and a for which $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{n+1}$.

Example. Take $f(x) = \sin(x)$ and $a = 0$. Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{(4)}(0) = 0$, $f^{(5)}(c) = \cos(c)$. So there is some c between 0 and x for which
$\sin(x) = 0 + 1 \cdot x^1 + 0 \cdot x^2 - \frac{1}{3!}x^3 + 0 \cdot x^4 + \frac{\cos(c)}{5!}x^5$.

In particular, $|\sin(x) - (x - \frac{1}{3!}x^3)| \leq \frac{1}{5!}|x|^5$. So (for instance) $\sin(0.1) \approx 0.1 - \frac{1}{3!}(0.1)^3$ with error less than 10^{-7}.

Taylor series

Example. Generalizing the preceding example, write $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + (-1)^{n+1}\frac{\cos(c)}{(2n+3)!}x^{2n+3}$ for some point c between 0 and x.

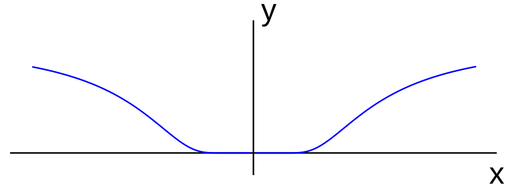
Because $|\cos(c)| \le 1$ for every c, and $\lim_{n\to\infty} x^{2n+3}/(2n+3)! = 0$, taking the limit gives $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, the *Taylor series* for $\sin(x)$ at 0.

A function that can be represented by a Taylor series in powers of $(x-a)$ is called *analytic* at a. The case $a = 0$ (illustrated above) is often called a *Maclaurin series*.

Some examples of analytic functions are polynomials, $\sin(x)$, $\cos(x)$, and e^x (for all x); also $1/(1-x)$ for $|x| < 1$.

A strange example

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$



This function is not analytic at 0. The Maclaurin series converges, but not to the function. In fact, every Maclaurin series coefficient is equal to 0.

$f'(0) = \lim_{x\to 0^+} \frac{e^{-1/x^2}}{x} = \lim_{t\to\infty} \frac{e^{-t^2}}{1/t} = \lim_{t\to\infty} \frac{t}{e^{t^2}} \overset{\text{l'Hôpital}}{=} \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$, and similarly for higher-order derivatives.

Homework

1. Read section 17.4, pages 236–238.

2. The third examination is scheduled for Wednesday, December 1.

One of the problems on the exam will be to prove a version of l'Hôpital's rule selected from the following eight possibilities:

$$\begin{aligned} & \{\tfrac{0}{0} \text{ or } \tfrac{\infty}{\infty}\} \\ \text{and} \quad & \{x \to a \text{ or } x \to \infty\} \\ \text{and} \quad & \{\lim \tfrac{f'(x)}{g'(x)} = L \text{ or } \lim \tfrac{f'(x)}{g'(x)} = \infty\}. \end{aligned}$$

Work on proofs of two cases (to discuss in class).