Advanced Calculus I

Instructions Solve **six** of the following seven problems. Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "determine" or "true/false" or "give an example" requires a supporting explanation. Please explain your reasoning in complete sentences.

1. If $x_1, x_2, \ldots$ is a Cauchy sequence of real numbers, is it necessarily true that $|x_1|, |x_2|, \ldots$ is a Cauchy sequence too? Give a proof or a counterexample, whichever is appropriate.

   Solution. The statement is true. For suppose ε is a specified positive real number. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then there exists a natural number N such that $|x_n - x_m| < \varepsilon$ when $n \ge N$ and $m \ge N$. By the triangle inequality,

   $$\big||x_n| - |x_m|\big| \le |x_n - x_m|,$$

   so $\big||x_n| - |x_m|\big|$ inherits the property of being less than ε when $n \ge N$ and $m \ge N$. Therefore $\{|x_n|\}_{n=1}^{\infty}$ is a Cauchy sequence.

2. (a) State the definition of what "$\lim_{x\to 0} f(x) = 0$" means.

   Solution. To every positive real number ε there corresponds a positive real number δ such that $|f(x)| < \varepsilon$ when $0 < |x| < \delta$.

   (b) Use the definition to prove that $\lim_{x\to 0} e^{-1/x^2} = 0$.

   Solution. Suppose ε is an arbitrary positive real number. Set δ equal to

   $$\frac{1}{\sqrt{\log\left(\frac{1+\varepsilon}{\varepsilon}\right)}}.$$

   If $|x| < \delta$, then

   $$x^2 < \delta^2 = \frac{1}{\log\left(\frac{1+\varepsilon}{\varepsilon}\right)},$$

   so if additionally $x \neq 0$, then

   $$-\frac{1}{x^2} < -\log\left(\frac{1+\varepsilon}{\varepsilon}\right) = \log\left(\frac{\varepsilon}{1+\varepsilon}\right),$$

and since the exponential function is increasing,

$$e^{-1/x^2} < \frac{\varepsilon}{1+\varepsilon} < \varepsilon.$$

Thus the definition of limit is satisfied.

Remark Setting δ equal to

$$\frac{1}{\sqrt{\log(1/\varepsilon)}}$$

almost works, except that $\log(1/\varepsilon)$ is negative when $\varepsilon > 1$. You could, however, assume without loss of generality that $\varepsilon < 1$ (since making $|f(x)|$ less than a number smaller than ε certainly ensures that $|f(x)|$ is less than ε).

3. Evidently $2^x = x^2$ when $x = 2$ and when $x = 4$. Are there any negative values of the real number x for which $2^x = x^2$? Explain how you know. [You may assume that 2^x is an everywhere differentiable function of x.]

   Solution. Set $f(x)$ equal to $2^x - x^2$. Then f is a continuous function, $f(0) = 1$, and $f(-1) = \frac{1}{2} - 1 = -\frac{1}{2}$. By the intermediate-value theorem, there must be a value of x between -1 and 0 for which $f(x) = 0$. This negative value of x has the property that $2^x = x^2$.

4. If $f(x) = \sin(x)$ for every real number x, is the function $f\colon \mathbf{R} \to \mathbf{R}$ uniformly continuous on $\mathbf{R}$? Explain why or why not.

   Solution. The function is uniformly continuous. One reason is that the derivative $f'(x)$ equals $\cos(x)$, which is a bounded function. We covered a theorem stating that a function with a bounded derivative is necessarily a uniformly continuous function. (In the present context, you could reprove that theorem as follows: if x and y are two arbitrary real numbers, then by the mean-value theorem there is a point c between x and y such that $|\sin(x) - \sin(y)| = |\cos(c)|\,|x - y| \le |x - y|$. So δ can be taken equal to ε in the definition of uniform continuity.)

   A second method is to invoke the theorem that a continuous function on a closed, bounded interval is automatically uniformly continuous.

Although the domain in this problem is *not* bounded, the function $\sin(x)$ is periodic, so it essentially lives on a closed, bounded interval. Here are more details for this approach.

Fix a positive real number ε. By the indicated theorem, there is a positive δ, which may be taken to be less than π, with the property that if x and y are points of the interval $[0, 3\pi]$ such that $|x - y| < \delta$, then $|\sin(x) - \sin(y)| < \varepsilon$. Now if $\tilde{x}$ and $\tilde{y}$ are arbitrary real numbers such that $|\tilde{x} - \tilde{y}| < \delta$, then there are numbers x and y in the interval $[0, 3\pi]$ such that $|\tilde{x} - \tilde{y}| = |x - y|$ and $\sin(\tilde{x}) = \sin(x)$ and $\sin(\tilde{y}) = \sin(y)$. (Simply translate the numbers $\tilde{x}$ and $\tilde{y}$ along the number line by $2\pi n$ for a suitable integer n to make the smaller of the two numbers lie in the interval $[0, 2\pi]$.) Then $|\sin(\tilde{x}) - \sin(\tilde{y})| = |\sin(x) - \sin(y)| < \varepsilon$.

5. Suppose that

$$f(x) = \begin{cases} x\cos(1/x), & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

Is the function f differentiable at the point where $x = 0$? Explain why or why not.

Solution. The function is continuous at the point where $x = 0$ by the sandwich theorem, but the function is not differentiable at the point where $x = 0$. Indeed,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x\cos(1/x) - 0}{x - 0} = \cos(1/x) \qquad \text{when } x \neq 0.$$

The derivative $f'(0)$ exists if and only if the preceding quantity has a limit as x approaches 0. But we saw in class that $\cos(1/x)$ does not have a limit as x approaches 0. (Explicitly, if $x_n = (n\pi)^{-1}$, then $x_n \to 0$, but $\cos(x_n) = (-1)^n$, and $(-1)^n$ has no limit as $n \to \infty$.)

Remark The product rule and the chain rule imply that

$$f'(x) = \cos\left(\frac{1}{x}\right) + \frac{1}{x}\sin\left(\frac{1}{x}\right) \qquad \text{when } x \neq 0,$$

so evidently $\lim_{x\to 0} f'(x)$ does not exist. This observation by itself shows only that the derivative is not continuous at 0; one cannot conclude without further analysis that $f'(0)$ fails to exist. Indeed, you know from Example 4.8 on page 103 that there is a similar function whose derivative is discontinuous yet exists everywhere.

6. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a differentiable function, and $\lim_{x\to\infty} f'(x) = 3$. Determine $\lim_{x\to\infty} \bigl(f(x+2) - f(x)\bigr)$.

   Solution. By the mean-value theorem, there is a point c_x between x and $x+2$ such that $f(x+2) - f(x) = f'(c_x)((x+2) - x) = 2f'(c_x)$. Now $c_x \to \infty$ when $x \to \infty$ (since $c_x > x$), so $f'(c_x) \to 3$ when $x \to \infty$. Therefore $f(x+2) - f(x) = 2f'(c_x) \to 6$ when $x \to \infty$.

7. Suppose $f(x) = \dfrac{2}{1+x}$ for every positive real number x, and let g denote the iterated composition $\underbrace{f \circ f \circ \cdots \circ f}_{409 \text{ copies of } f}$. Determine the derivative $g'(1)$.

   Solution. Observe that $f(1) = 1$. Consequently, by the chain rule, $(f \circ f)'(1) = f'(f(1))f'(1) = f'(1)^2$. It follows by a straightforward induction argument that $g'(1) = f'(1)^{409}$. You know from elementary calculus (by the quotient rule, for example) that $f'(1) = -1/2$. Therefore $g'(1) = -1/2^{409}$.

   Remark The large number 409 is a hint that there must be a way to solve the problem without computing an explicit formula for $g(x)$.