Advanced Calculus I

A Definitions and examples

1. Continuity

   (a) State the definition of "$f\colon (0,1) \to \mathbf{R}$ is continuous".

   Solution. See Definition 3.19 on page 83.

   (b) Give a concrete example of a continuous function.

   Solution. Some examples are constant functions, polynomials, the sine function, the cosine function, and the exponential function.

   (c) Give a concrete example of a function that is not continuous.

   Solution. Step functions and the Dirichlet function are possible examples.

2. Differentiability

   (a) State the definition of "$f\colon (0,1) \to \mathbf{R}$ is differentiable".

   Solution. See Definitions 4.1 and 4.6 on pages 98 and 102.

   (b) Give a concrete example of a function that is differentiable.

   Solution. Some examples are constant functions, polynomials, the sine function, the cosine function, and the exponential function.

   (c) Give a concrete example of a function that is not differentiable.

   Solution. Some examples are the absolute-value function $|x|$ on the interval $(-1,1)$, the shifted function $|x-\frac{1}{2}|$ on the interval $(0,1)$, and the discontinuous functions from the previous problem.

3. Integrability

   (a) State the definition of "$f\colon [0,1] \to \mathbf{R}$ is Riemann integrable".

   Solution. See Definition 5.9 on page 134.

(b) Give a concrete example of a function that is Riemann integrable.

Solution. Constant functions, polynomials, the sine function, the cosine function, and the exponential function are some examples of integrable functions on bounded intervals.

(c) Give a concrete example of a function that is not Riemann integrable.

Solution. One example is the Dirichlet function. Another example is any unbounded function, say

$$\begin{cases} 1/x, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

B Theorems and proofs

Here are some of the important theorems from the course:

- Bolzano–Weierstrass theorem
- Intermediate-value theorem
- Mean-value theorem
- Taylor's formula
- l'Hôpital's rule
- Fundamental theorem of calculus

4. Give careful statements of *three* of the indicated theorems. (For a theorem that has several versions, state any one version.)

5. Prove *one* of the indicated theorems. (For a theorem that has several versions, prove any one version.)

Solution. The indicated theorems (with proofs) are in the textbook as Theorem 2.26 on page 56, Theorem 3.29 on page 87, Theorem 4.15 on page 111, Theorem 4.24 on page 117, Theorem 4.27 on page 120, and Theorem 5.28 on page 152.

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C Problems

Solve *two* of the following four problems.

6. Prove that $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for every natural number n.

   Solution. The proof is by induction on n. When $n = 1$, both sides equal 1, so the basis step of the induction argument is valid.

   Suppose, then, that the equation is known to hold for a certain natural number n. It follows by adding $(n+1)^3$ to both sides that

   $$\sum_{k=1}^{n+1} k^3 = (n+1)^3 + \frac{n^2(n+1)^2}{4}.$$

   Routine algebra shows that the right-hand side simplifies as follows:

   $$\begin{aligned}(n+1)^3 + \frac{n^2(n+1)^2}{4} &= (n+1)^2 \left[(n+1) + \frac{n^2}{4}\right] \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4}.\end{aligned}$$

   Consequently, if the indicated equation holds for a certain natural number n, then the equation holds for the successor number.

   By induction, the equation holds for every natural number.

7. Prove that $\left\{n \sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence.

   Solution. The indicated sequence is a sequence of real numbers, so an equivalent statement is that the sequence has a limit. You know from class or by l'Hôpital's rule that

   $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

   Letting x approach 0 along the sequence $\{1/n\}_{n=1}^{\infty}$ shows that

   $$\lim_{n \to \infty} \frac{\sin(1/n)}{1/n} = 1, \qquad \text{or} \qquad \lim_{n \to \infty} n \sin(1/n) = 1.$$

   Thus the indicated sequence not only converges but has limit 1.

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8. Define $f\colon (0,\infty) \to (0,\infty)$ by setting $f(x)$ equal to xe^x for each positive real number x. Prove that f has an inverse function, and evaluate the derivative $(f^{-1})'(e)$.

   Solution. Since $f'(x) = xe^x + e^x > 0$ when $x > 0$, the function f is strictly increasing, hence one-to-one. Moreover, $\lim_{x\to\infty} f(x) = \infty$, and $\lim_{x\to 0} f(x) = 0$. By the intermediate-value theorem, the range of the continuous function f is all of $(0,\infty)$. Being both one-to-one and onto, the function f has an inverse.

   By inspection, $f(1) = e$, so the theorem about differentiating an inverse function shows that

$$(f^{-1})'(e) = \frac{1}{f'(1)} = \frac{1}{2e}.$$

9. Let a_n equal $\displaystyle\int_1^n \frac{\sin(x)}{\sqrt{x}}\,dx$ for each natural number n. Prove that $\lim_{n\to\infty} a_n$ exists.

   Solution. The key idea is to integrate by parts:

$$a_n = \frac{-\cos(n)}{\sqrt{n}} + \cos(1) - \frac{1}{2}\int_1^n \frac{\cos(x)}{x^{3/2}}\,dx.$$

   Now $|-\cos(n)/\sqrt{n}\,| \le 1/\sqrt{n} \to 0$ as $n \to \infty$, so the first term on the right-hand side has a limit by the sandwich theorem. The second term is constant, so what remains to show is that

$$\lim_{n\to\infty} \int_1^n \frac{\cos(x)}{x^{3/2}}\,dx \qquad \text{exists,}$$

   or equivalently that

$$\left\{ \int_1^n \frac{\cos(x)}{x^{3/2}}\,dx \right\}_{n=1}^{\infty} \qquad \text{is a Cauchy sequence.}$$

   If $m < n$, then

$$\left| \int_1^n \frac{\cos(x)}{x^{3/2}}\,dx - \int_1^m \frac{\cos(x)}{x^{3/2}}\,dx \right| = \left| \int_m^n \frac{\cos(x)}{x^{3/2}}\,dx \right| \le \int_m^n \left| \frac{\cos(x)}{x^{3/2}} \right| dx$$
$$\le \int_m^n \frac{1}{x^{3/2}}\,dx = 2\left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{n}} \right).$$

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If a positive ε is prescribed, then the right-hand side will certainly be less than ε when $m > 4/\varepsilon^2$. Consequently, the indicated sequence of integrals is a Cauchy sequence.