Final Examination

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B, and optionally the extra-credit question 10 in Part D.

Students in Section 200 should answer questions 1–3 in Part A and questions 7–9 in Part C, and optionally the extra-credit question 10 in Part D.

Part A, for both Section 501 and Section 200

1. Consider a function f defined as follows:

$$f(x) = 409x - x^{409} \qquad \text{when} \qquad 0 < x < 2.$$

Does the function f attain a maximum on the open interval $(0, 2)$? What about a minimum? Why or why not?

2. Consider a sequence $\{x_n\}$ of real numbers defined recursively as follows:

$$x_1 = 1, \qquad \text{and} \qquad x_{n+1} = \frac{409 + x_n}{2} \quad \text{when} \quad n \geq 1.$$

Does $\lim_{n\to\infty} x_n$ exist? Explain why or why not.

3. Consider the following three properties that a function f might or might not have:

(C) f is continuous on the interval $[0, 2]$.

(D) f is differentiable on the interval $[0, 2]$.

(I) f is Riemann integrable on the interval $[0, 2]$.

There are six possible implications between these properties: (C) $\Rightarrow$ (D); (D) $\Rightarrow$ (C); (C) $\Rightarrow$ (I); (I) $\Rightarrow$ (C); (D) $\Rightarrow$ (I); (I) $\Rightarrow$ (D). Which implications are valid? Why? For the implications that are not valid, give counterexamples.

Part B, for Section 501 only

4. State the following theorems:

a) l'Hôpital's rule (some version);

b) the fundamental theorem of calculus (relating derivatives and integrals);

c) some (other) theorem from this course that involves the concept of compactness.

5. Prove that $2^n \geq 2n$ for every natural number n.

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6. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to\infty} f(x) = 0$, yet $\lim_{x\to\infty} \int_1^x f(t)\,dt = \infty$.

Part C, for Section 200 only

7. State the following theorems:
   a) some theorem from this course named after a European mathematician whose name begins with the letter "C";
   b) some (other) theorem from this course that involves the concept of a covering by open sets or by closed sets;
   c) some (other) theorem from this course that involves the intermediate-value property (Darboux property).
8. Prove that $2^n \geq n^2$ for every natural number n larger than 3.
9. Give an example of a bounded function f on the interval $[0, 1]$ such that $\lim_{x\to0+} f(x)$ does not exist, yet $\lim_{x\to0+} \int_x^1 f(t)\,dt$ does exist.

Part D, optional extra-credit question for both Section 200 and Section 501

10. Write an essay on the following topic: What is the most important concept or principle or theorem from this course? Why?