# Examination 1

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. State one of the following: the Archimedean property of the real numbers; the Bolzano–Weierstrass theorem; Cauchy's criterion for convergence.

2. Suppose $A$ and $B$ are bounded, non-empty sets of real numbers, and let $C$ denote the union $A \cup B$. Show that $\sup C$ equals the maximum of the two numbers $\sup A$ and $\sup B$.

3. Give an example of a set having at least one boundary point that is not an accumulation point and also at least one accumulation point that is not a boundary point. Explain why your example has the required properties.

4. Determine the smallest natural number $k$ with the property that

$$0.999 < \frac{n^2 - 1}{n^2 + 1} < 1.001 \qquad \text{for every natural number } n \text{ exceeding } 10^k.$$

5. Suppose $E$ is a compact set of real numbers and $F$ is a closed set. Is the intersection $E \cap F$ necessarily compact? Give either a proof or a counterexample, as appropriate.

6. Consider the sequence defined recursively as follows:

$$x_1 = 1, \qquad \text{and} \qquad x_{n+1} = \log(1 + x_n) \qquad \text{when } n \geq 1.$$

   Here "log" means the natural logarithm function (which is often called "ln" in elementary mathematics). Say as much as you can about the value of $\limsup_{n\to\infty} x_n$ for this sequence.

   Hint: Use the following diagram, which shows that the expression $\log(1+x)$ is an increasing function of $x$ whose graph is concave down. The tangent line at the origin has slope 1.