Announcements

- No office hour on March 31 (Friday) or April 3 (Monday). I will be away from campus.
- Reminder: The second examination takes place in class on April 11 (Tuesday).

Properties of continuous functions on intervals

If I is an interval and $f\colon I \to \mathbb{R}$ is continuous at every point of I, then

1. the image $f(I)$ is an interval (possibly degenerate, possibly unbounded) [intermediate-value theorem];
2. if I is a *compact* interval $[a, b]$, then the image $f(I)$ is a *compact* interval (possibly a single point) [extreme-value theorem];
3. f is injective if and only if f is strictly monotonic.

Comparing notions of function and injective function

$f\colon D \to R$
Function means $\forall x \in D\ \exists! y \in R$ such that $f(x) = y$.

Injective means $\forall y \in R$ there is at most one $x \in D$ such that $f(x) = y$.

Bijective means $\forall y \in R\ \exists! x \in D$ such that $f(x) = y$.

Proof of the intermediate-value theorem

Hypotheses. $f : I \to \mathbb{R}$ is continuous at all points of the interval I, and a and b are two points of I, with $a < b$.
Conclusion. Every number y between $f(a)$ and $f(b)$ is in the image of f.

Proof.

Let S denote $\{ x \in [a, b] : f(x) \leq y \}$. Then S is non-empty (because either $a \in S$ or $b \in S$) and bounded above (by b). By the completeness property of the real numbers, there is a least upper bound, say c. The goal is to show that $f(c) = y$.
WLOG suppose $a < c < b$. Consider a sequence (x_n) approaching c from the right. By continuity, $f(x_n) \to f(c)$. But $x_n \notin S$, so $f(x_n) > y$. Limits preserve *weak* inequalities, so the limit $f(c)$ is $\geq y$.
Take a new sequence (z_n) such that $z_n \in S$ and $z_n \to c$ (this sequence exists by definition of supremum). Again by continuity, $f(z_n) \to f(c)$. And $f(z_n) \leq y$ because $z_n \in S$. So the limit $f(c)$ is $\leq y$.

□

Assignment due next class

Exercises 1 and 3 on page 91 in Section 5.4.