Reminder

The second examination takes place in class on Tuesday, April 11.

The exam covers an open subset of Chapters 4–7.

Please bring your own paper to work on.

Exercise on continuity

Suppose $f\colon [0,1] \to \mathbb{R}$ is a continuous function.

1. How many additional properties of f can you deduce?
2. How many theorems about f can you state?

Math 220 revisited

The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is equipped with three binary operations:

- addition $f + g$,
- multiplication $f \times g$,
- composition $f \circ g$.

For each operation, does it have

1. the commutative property?
   $+$ yes, $\times$ yes, $\circ$ no
2. the associative property?
   $+$ yes, $\times$ yes
   $(f \circ g) \circ h(x) = f \circ (g \circ h)(x)$?
   Yes, both sides equal $f(g(h(x)))$.
3. an identity element?
   yes for all three

Limits of sequences of continuous functions

Example

Suppose $f_n(x) = \dfrac{x^{2n}}{1 + x^{2n}}$ for every real number x. Then f_n is continuous. Why?

Continuous functions are preserved by addition, multiplication, and division (as long as we don't divide by zero).

Suppose $f(x) = \lim_{n\to\infty} f_n(x)$ for every real number x. Then f has points of discontinuity. Why?

$$f(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ 1/2, & \text{if } x = \pm 1, \\ 1, & \text{if } |x| > 1. \end{cases}$$

The limit of continuous functions is not necessarily continuous.

Pointwise convergence versus uniform convergence

$f_n \to f$ pointwise means
$\forall x \; \forall \varepsilon \; \exists N$ such that $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$.
The cutoff N depends on both x and ε.

$f_n \to f$ uniformly means
$\forall \varepsilon \; \exists N$ such that $\forall x \;\; n \geq N \implies |f_n(x) - f(x)| < \varepsilon$.
The cutoff N depends on ε but not on x.

Example of non-uniform convergence