Exam grading

Score computation: 50-point baseline plus 10 points per problem.

Some properties of the derivative

- Differentiation is a *linear* operation.
  The derivative of a sum is the sum of the derivatives, and the derivative of a constant times f is the constant times f'.
- A differentiable function is necessarily continuous, but a continuous function need not be differentiable.
  Indeed, if f is differentiable at a point c, then there exists a function A, continuous at c, such that
  $f(x) = A(x)(x - c) + f(c)$, and the right-hand side is continuous at c.
  Example: $|x|$ is continuous but not differentiable at 0.

Karl Weierstrass constructed (1872) a continuous function that has a derivative at no point.

Exercise

Can you find a function f, differentiable at 0, such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}$$

for each natural number n?
By continuity, $f(0)$ has to be 0.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{n \to \infty} \frac{\frac{(-1)^n}{n}}{\frac{1}{n}} = \lim_{n \to \infty} (-1)^n$$

does not exist!
No such function f exists.

What goes up must come down

Theorem (Rolle's theorem)

If $f : [a, b] \to \mathbb{R}$ *is differentiable, and* $f(a) = f(b)$*, then there is some point* c *(at least one) inside the interval* (a, b) *such that* $f'(c) = 0$*.*

Proof.

Since f is, in particular, continuous, the extreme-value theorem implies that f attains a maximum value and also attains a minimum value. If the max and the min both occur at endpoints, then f is a constant function, so the derivative is identically zero. The interesting case is when the max or the min is taken at an interior point. WLOG suppose the max occurs at an interior point c.
If $x > c$, then

$$\frac{f(x) - f(c)}{x - c} \leq 0 \qquad \text{so } f'(c) \leq 0.$$

□

continuation

Similarly, if $x < c$, the same argument shows that $f'(c) \geq 0$.
By hypothesis, the derivative at c exists as a two-sided limit, so since $f'(c) \leq 0$ and $f'(c) \geq 0$, we conclude that $f'(c) = 0$.

Theorems on average rate of change

Theorem (Basic version of the mean-value theorem)

If $f : [a, b] \to \mathbb{R}$ is differentiable, then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem (Cauchy's version of the mean-value theorem)

If f and g are two differentiable functions on $[a, b]$, then there is a point c in (a, b) such that $g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$.

The second theorem is essentially the first theorem for a curve described by parametric equations.

Assignment for next time

Exercise 7 on page 133 in §8.5 (a version of l'Hôpital's rule).