Final Examination

Instructions. Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. State three definitions or theorems that use the phrase "for every positive ε."

2. For each part, give an example of a non-empty subset S of $\mathbb{R}$ that satisfies the property.
   a) The set S has infinitely many limit points and also has empty interior.
   b) The set S is bounded, and $\sup\{\, x^2 : x \in S \,\} \neq (\sup S)^2$.
   (As usual, the notation "sup" means the supremum, that is, the least upper bound.)

3. Consider the sequence defined recursively as follows:
$$x_1 = \sin(1), \qquad \text{and} \qquad x_{n+1} = \sin(x_n) \quad \text{when } n \geq 1.$$
   Does this sequence of real numbers converge? Explain why or why not.
   (You may assume the standard properties of the sine function shown on the second page.)

4. Here are five concepts from this course commencing with the consonant c:
   a) closure of a set
   b) compact set
   c) completeness axiom
   d) continuous function
   e) covering of a set

   Explain the meaning of **three** of these concepts.

5. When E is a non-empty subset of $\mathbb{R}$, the distance-to-E function d_E is defined as follows:
$$d_E(x) = \inf\{\, |x-y| : y \in E \,\} \qquad \text{for each real number } x.$$
   Prove that if the set E is closed, then "inf" can be replaced by "min" in this definition: in other words, for each x, the infimum (greatest lower bound) is attained for some y in E.

6. When $f : [0,2] \to \mathbb{R}$ is a monotonic, differentiable function for which $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$, what (if anything) can be deduced about
   a) the limit $\lim_{x\to 1} f(x)$?
   b) the derivative $f'(1)$?
   c) the integral $\int_0^2 f(x)\,dx$?

   Explain your reasoning.

Extra Credit Problem. Suppose $f : (-1,1) \to \mathbb{R}$ is defined as follows: $f(x) = \dfrac{x^{409}+1}{x^{501}+1}$. Say as much as you can about the image, the set $\{\, f(x) : -1 < x < 1 \,\}$.

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Background for Problem 3. You may assume as standard knowledge both the graph below and the fact that $\frac{d}{dx}\sin(x) = \cos(x) = \sin\left(\frac{1}{2}\pi - x\right)$.