Examination 2

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Give an example of a function, defined on the closed interval $[0, 1]$, that attains a maximum but does not have the intermediate-value property (Darboux property).

Solution. A continuous function on an interval always has the intermediate-value property, so every solution to this problem must have at least one point of discontinuity. One example is a step function, say

$$f(x) = \begin{cases} 0, & \text{if } x < 1/2, \\ 1, & \text{if } x \geq 1/2. \end{cases}$$

The maximum is 1, which evidently is attained, but the function fails to take on the values that lie between 0 and 1.

Another example is the function mentioned in problem 6 below with x_0 taken equal to 0; this function is the characteristic function of the singleton set $\{0\}$. More generally, if E is an arbitrary nonvoid proper subset of the interval $[0, 1]$, then the characteristic function χ_E is an example. The example $\chi_{\mathbb{Q}}$ lacks the intermediate-value property on every interval. There are many other examples.

2. Suppose f is a twice-differentiable function. Determine

$$\lim_{x \to 1} \frac{f(f(x)) - f(x)}{f(x) - 1}$$

given the information in the following table.

x	$f(x)$	$f'(x)$	$f''(x)$
0	2	1	0
1	1	0	2
2	0	2	1

Solution. Method 1. Since $f(1) = 1$, also $f(f(1)) = 1$. The function f is certainly continuous (being differentiable), so both the numerator of the fraction and the denominator of the fraction have limit equal to 0. Since the derivative of the composite function $f \circ f$ at a general point x equals the product $f'(f(x))f'(x)$, l'Hôpital's rule implies that the limit equals

$$\lim_{x \to 1} \frac{f'(f(x))f'(x) - f'(x)}{f'(x)}$$

if the latter limit exists. If $f'(x)$ is nonzero in a punctured neighborhood of the point 1, then the term $f'(x)$ can be divided out of both the numerator and the denominator, reducing the problem to the new limit

$$\lim_{x \to 1} f'(f(x)) - 1.$$

Since f is twice differentiable, the first derivative f' is continuous, so the answer is

$$f'(f(1)) - 1, \qquad \text{or} \qquad f'(1) - 1, \qquad \text{or} \qquad -1.$$

Is l'Hôpital's rule actually applicable? The rule has a hypothesis that both the denominator $f(x) - 1$ and its derivative $f'(x)$ should be nonzero in some punctured neighborhood of the point 1; this hypothesis needs to be verified.

Since $f''(1)$ has the positive value 2, and $f'(1)$ equals 0, the first derivative f' is positive in some right-hand neighborhood of the point 1 and negative in some left-hand neighborhood of 1. (Compare Exercise 7.2.12 that you solved in a homework assignment.) Thus there does exist a punctured neighborhood of 1 in which the first derivative f' is nonzero, as required. Moreover, the information about the first derivative implies that f is decreasing in some left-hand neighborhood of 1 and increasing in some right-hand neighborhood of 1, so the function $f(x) - 1$ has a strict local minimum when $x = 1$. This minimum value is 0, so $f(x) - 1$ is nonzero in some punctured neighborhood of the point 1, as required.

In conclusion, the hypotheses of l'Hôpital's rule are satisfied, so the computed value of -1 for the limit is valid.

Method 2. The expression has the form

$$\frac{f(y) - y}{y - 1}, \qquad \text{or} \qquad \frac{f(y) - 1}{y - 1} - 1,$$

where $y = f(x)$. Since f is continuous, the value of y tends to 1 when x tends to 1, so the original limit must be equal to

$$\lim_{y \to 1} \frac{f(y) - 1}{y - 1} - 1.$$

By the definition of derivative, this expression equals $f'(1) - 1$, or -1.

As in the first solution, this method requires knowing that there is a neighborhood of 1 such that $f(x) - 1 \neq 0$ when x lies in the neighborhood and is different from 1. If there were no such neighborhood, then there would be a sequence $\{x_n\}$ of points different from 1 and converging to 1 such that $f(x_n) = 1$. By Rolle's theorem, there would be a corresponding sequence $\{c_n\}$ converging to 1 such that $f'(c_n) = 0$. Then $f''(1)$ would be equal to the limit

$$\lim_{n \to \infty} \frac{f'(c_n) - f'(1)}{c_n - 1}.$$

But the numerator equals 0, and $f''(1)$ equals 2, which is a contradiction. Thus $y - 1$ is indeed nonzero when $y = f(x)$ and x is close to but different from 1. So the indicated computation of the original limit is valid.

3. State one of the following items (extra credit for correctly stating both):

   a) Cauchy's version of the mean-value theorem; or

b) the definition of what it means for a function f to be uniformly continuous on an interval.

Solution. The first part is Theorem 7.21, and the second part is Definition 5.46.

4. A function $f(x)$ taking only positive values is called *logarithmically convex* when $\log f(x)$ is a convex function.

   (Here log denotes the natural logarithm function, but the definition is actually independent of the base of the logarithm as long as the base is greater than 1.)

   Show that if $f(x)$ is logarithmically convex, then $f(x)$ must be convex. You may assume that $f(x)$ is twice differentiable.

Solution. Method 1. The hypothesis is that if x_1 and x_2 are points in the domain of f, and q_1 and q_2 are positive numbers whose sum is 1, then

$$\log f(q_1 x_1 + q_2 x_2) \le q_1 \log f(x_1) + q_2 \log f(x_2) = \log \left(f(x_1)^{q_1} f(x_2)^{q_2} \right).$$

The exponential function is increasing and so preserves inequalities. Therefore exponentiating shows that

$$f(q_1 x_1 + q_2 x_2) \le f(x_1)^{q_1} f(x_2)^{q_2}.$$

As proved in class, the geometric mean is less than or equal to the arithmetic mean, so the right-hand side is at most $q_1 f(x_1) + q_2 f(x_2)$. Therefore f satisfies the inequality that characterizes convexity.

Method 2. When f is twice differentiable, convexity of $\log f$ is equivalent to the second derivative being bigger than or equal to 0. Now the first derivative of $\log f$ is f'/f, so the second derivative is $(ff'' - (f')^2)/f^2$. Saying that this expression is nonnegative is the same as saying that $ff'' \ge (f')^2$. The right-hand side is a square, hence is nonnegative, so the inequality implies that $ff'' \ge 0$. But f is positive by hypothesis, so $f'' \ge 0$. Accordingly, the function f is convex.

Remark. The solution shows that a logarithmically convex function is convex. On the other hand, the converse is not valid. For example, the function x^2 is convex and positive when $x > 0$, but $\log x^2$ is not convex: indeed, $\log x^2 = 2 \log x$, and $\log x$ has negative second derivative.

5. Prove one of the following statements (extra credit for proving both):

   a) If f is a differentiable function on the interval $(0, 1)$ such that $f(x)f'(x) = 0$ for every value of x, then f must be a constant function.

   Solution. Method 1. Observe that $f(x)f'(x)$ is half the derivative of $(f(x))^2$. The hypothesis implies that the derivative of $(f(x))^2$ is identically equal to 0, so $(f(x))^2$

is a constant function. If the constant value is 0, then $f(x)$ is the 0 function. If the constant value is some nonzero c (necessarily positive), then at each x the value of $f(x)$ is either $\sqrt{c}$ or $-\sqrt{c}$, the choice possibly depending on x. But f is differentiable, hence continuous, so f has the intermediate-value property. If f takes both values $\sqrt{c}$ and $-\sqrt{c}$, then f must take all the intermediate values, which is a contradiction. Therefore $f(x)$ is either constantly equal to $\sqrt{c}$ or constantly equal to $-\sqrt{c}$.

Method 2. If $f(x)$ is identically equal to 0, then there is nothing to prove. Suppose, then, that there is some value x_0 such that $f(x_0) \neq 0$. Since f is continuous (because differentiable), there is a neighborhood of x_0 such that $f(x) \neq 0$ for every x in that neighborhood. But $f(x)f'(x)$ is identically equal to 0 in the neighborhood, so $f'(x)$ must be identically equal to 0 in the neighborhood. Therefore $f(x)$ is identically equal to the constant value $f(x_0)$ in the neighborhood.

Let s denote the supremum of the set of x such that the function f is identically equal to $f(x_0)$ on the interval (x_0, x). If s is strictly less than 1, then $f(s)$ is defined and is equal to $f(x_0)$ by continuity of f. By the argument in the preceding paragraph, the function f is constantly equal to $f(x_0)$ in some neighborhood of s, contradicting the definition of supremum. Therefore s must be equal to 1, and f is constantly equal to $f(x_0)$ on the interval $(x_0, 1)$. A symmetric argument using the infimum shows that f is constantly equal to $f(x_0)$ on the interval $(0, x_0)$. Thus f is constant on the whole interval $(0, 1)$, as claimed.

Method 3, added April 13. A student found the following nice proof.

If $f'(x)$ is identically equal to 0, then $f(x)$ is constant (by Theorem 7.24(v), which is an application of the mean-value theorem). So the required conclusion holds.

The other possibility is that at least one point x_0 exists for which $f'(x_0) \neq 0$. Without loss of generality, suppose that $f'(x_0) > 0$ (replace f by $-f$ if necessary).

Since the derivative is positive at x_0, there is a positive δ such that $f(x_0) < f(x)$ when $x_0 < x < x_0 + \delta$. Let x_1 be a point between x_0 and $x_0 + \delta$, and apply the mean-value theorem to deduce the existence of a value c between x_0 and x_1 such that

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(c).$$

By construction, both the numerator and the denominator of the fraction are positive, so $f'(c) > 0$.

The hypothesis of the problem implies that $f(c)f'(c) = 0$, but $f'(c) > 0$, so $f(c) = 0$. Since $x_0 < c < x_1 < x_0 + \delta$, the preceding paragraph implies that $f(x_0) < f(c)$. Thus $f(x_0)$ is a negative value. But $f'(x_0) > 0$ by the assumption in the second paragraph, so $f(x_0)f'(x_0) < 0$. This conclusion contradicts the hypothesis of the problem.

The contradiction means that the derivative f' must be identically equal to 0 after all, so f is constant, as observed in the first paragraph.

Examination 2

b) If $f : (0, 1) \to (0, 1)$ and $g : (0, 1) \to (0, 1)$ are two uniformly continuous functions, then the composite function $f \circ g$ must be uniformly continuous.

Solution. Fix an arbitrary positive ε. Since f is uniformly continuous, there is a positive δ such that $|f(y_1) - f(y_2)| < \varepsilon$ whenever y_1 and y_2 are in the domain of f and $|y_1 - y_2| < \delta$. Since g is uniformly continuous, there is a positive γ such that $|g(x_1) - g(x_2)| < \delta$ whenever x_1 and x_2 are in the domain of g and $|x_1 - x_2| < \gamma$. Accordingly, if $|x_1 - x_2| < \gamma$, then setting y_1 equal to $g(x_1)$ and y_2 equal to $g(x_2)$ shows that

$$|f \circ g(x_1) - f \circ g(x_2)| = |f(y_1) - f(y_2)| < \varepsilon.$$

Thus the composite function $f \circ g$ is uniformly continuous.

6. A function f taking values in the real numbers is called *upper semicontinuous* at a point x_0 of its domain if $\limsup\limits_{x \to x_0} f(x) \le f(x_0)$.

(Recall that lim sup denotes the largest limit that can be obtained along some sequence. One example of an upper semicontinuous function that fails to be continuous is the function that equals 0 when $x \ne x_0$ and equals 1 when $x = x_0$.)

Prove that if f is upper semicontinuous at every point of a closed, bounded interval $[a, b]$, then f is necessarily bounded above on the interval.

Solution. Method 1. Seeking a contradiction, suppose that f is not bounded above. Then there is a sequence $\{x_n\}$ in the interval $[a, b]$ such that $f(x_n) > n$. In view of the Bolzano–Weierstrass theorem, there is a subsequence $\{x_{n_k}\}$ that converges to some point x^* in the interval $[a, b]$. Now $\lim_{k\to\infty} f(x_{n_k}) = +\infty$ by construction, so $\limsup_{x\to x^*} f(x) = +\infty$, but $f(x^*)$ is some real number, so the definition of upper semicontinuity at x^* is contradicted. The contradiction shows that f must be bounded above after all.

Method 2. The definition of lim sup implies that each point x_0 in $[a, b]$ has a neighborhood on which f is bounded above by $1 + f(x_0)$. Since the interval $[a, b]$ is compact, the Heine–Borel property implies that there are finitely many points $x_1, \dots, x_n$ such that the corresponding neighborhoods cover the interval $[a, b]$. Consequently, the function f is bounded above by $1 + \max\{f(x_1), \dots, f(x_n)\}$.

Remark. A little more work shows that an upper semicontinuous function on a compact set attains a maximum.

There is a corresponding property of lower semicontinuity: namely, $\liminf\limits_{x \to x_0} f(x) \ge f(x_0)$. Continuous functions are the ones that are simultaneously upper semicontinuous and lower semicontinuous.