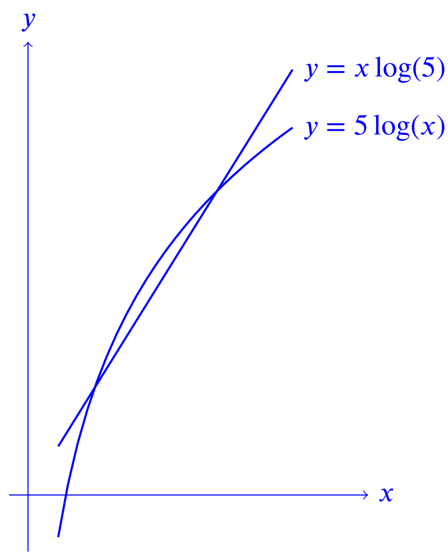
Final Examination

Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Show that the positive values of x for which $x \log(5) = 5 \log(x)$ are like the Sith Lords in *Star Wars*: "two there are—no more, no less."

Solution. The intuitive idea is that the graph of the function on the right-hand side is concave, and the graph of the function on the left-hand side is a line, so the intersection of the two curves must be either two points (the expected case) or one point (in the case of tangency) or no points (if the curves fail to meet). See the figure.



Some analysis is needed to confirm the intuition. Here is one way to proceed.

Let $f(x)$ denote the difference $x \log(5) - 5 \log(x)$. The goal is to show that $f(x) = 0$ for exactly two values of x. One method is to solve two subproblems: (a) there are at most two values of x for which $f(x) = 0$, and (b) there are at least two values of x for which $f(x) = 0$.

Since $f'(x) = \log(5) - \frac{5}{x}$, the derivative is equal to zero if and only if $x = 5/\log(5)$. Rolle's theorem implies that the derivative is equal to zero at least once between each two points where f is equal to zero. Since f' is equal to zero only once, there can be at most two points where f is equal to zero.

Evidently $f(5) = 0$, and $f'(5) = \log(5) - 1 > 0$ (since $5 > e$). Therefore $f(x)$ is increasing near the point where $x = 5$, so $f(x)$ is negative for values of x slightly smaller than 5. Now $f(1) = \log(5) > 0$, so the intermediate-value theorem for continuous functions implies that $f(x)$ is equal to zero for some value of x in the open interval $(1, 5)$. Thus $f(x)$ is equal to zero for at least two values of x: one value is 5, and a second value is somewhere between 1 and 5.

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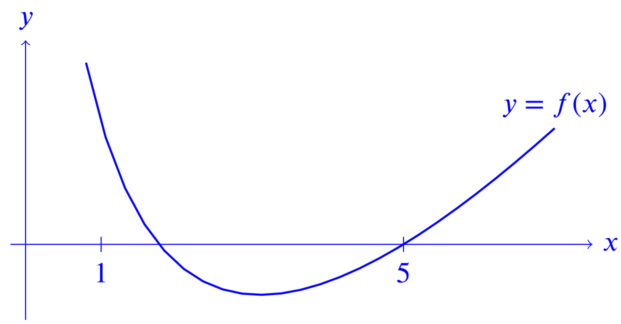
The deductions in the two preceding paragraphs combine to show that $f(x)$ is equal to zero for exactly two values of x.

Notice that $f''(x) = 5/x^2 > 0$, so the graph of f is convex (but this information is not needed). Moreover, $f(x) \to \infty$ when $x \to 0^+$ and also when $x \to \infty$ (but this information is not needed either). See the figure below.



2. Consider the sequence $\{x_n\}$ defined recursively as follows:

$$x_1 = 5, \qquad \text{and} \qquad x_{n+1} = 5 - \frac{1}{x_n} \quad \text{when } n \geq 1.$$

Does this sequence converge? Why or why not?

Solution. The sequence does converge, by the monotone convergence theorem. What needs to be shown is that the sequence is bounded and monotonic. Both properties can be established by induction.

The first claim is that all the terms of the sequence lie between 4 and 5. The basis step is evident, since $x_1 = 5$. If, for some natural number n, the value of x_n is known to lie between 4 and 5, then $1/x_n$ lies between $1/5$ and $1/4$, so $5 - 1/x_n$ lies between 4.75 and 4.8. Thus x_{n+1} lies between 4 and 5, so the induction step holds.

The second claim is that the sequence is decreasing, that is, $x_{n+1} < x_n$ for every natural number n. The basis step is evident, since $x_2 = 4.8 < 5 = x_1$. If, for some natural number n, the value of x_{n+1} is known to be less than x_n, then $1/x_{n+1}$ is greater than $1/x_n$, so $5 - 1/x_{n+1}$ is less than $5 - 1/x_n$, which means that x_{n+2} is less than x_{n+1}. Thus the induction step holds.

In summary, mathematical induction implies that the sequence is both decreasing and bounded below by 4, so the sequence converges to the infimum, which is some value no smaller than 4.

Remark. The question does not ask for the value of the limit, but that value is computable. Once the limit L is known to exist, passing to the limit in the recursive formula shows that $L = 5 - 1/L$, or $L^2 - 5L + 1 = 0$. This quadratic equation has two solutions, and L must be the one of those two solutions that is between 4 and 5: namely, $(5 + \sqrt{21})/2$.

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3. Let E denote the subset of the interval $[0.05, 0.55]$ consisting of real numbers that can be written as decimal expansions using only the digits 0 and 5. Explain why E is a compact set with empty interior.

Solution. A set of real numbers is compact if and only if the set is simultaneously closed and bounded. Since E is specified to be bounded both below (by 0.05) and above (by 0.55), what needs to be shown is that E is closed, that is, contains every accumulation point of E.

Observe that the smallest element of E having decimal expansion starting with a digit 5 is $0.5000\dots$, and the largest element of E starting with a digit 0 is $0.0555\dots$, so two elements of E that differ by less than 0.4 must have the same first decimal digit. Similarly, if two elements of E differ by less than 4×10^{-k}, then they share the first k decimal digits.

Accordingly, if a sequence $\{x_n\}$ of points of E converges, then for every natural number k, the kth decimal digit of x_n eventually stabilizes and so has a limit as $n \to \infty$, say a_k. Let x^* be the decimal expansion whose kth digit is a_k for each natural number k. Then $\lim_{n\to\infty} x_n = x^*$, and $x^* \in E$. Thus E contains every accumulation point, as required.

What remains to show is that E has no interior points or, equivalently, that every point of E is a limit of points lying in the complement of E. If x is an arbitrary point of E, then let x_n be the same decimal expansion except with the nth decimal digit changed to a 2. Then $x_n \notin E$, and $\lim_{n\to\infty} x_n = x$. Thus x is a boundary point of E, as required.

4. Show that $\displaystyle\lim_{n\to\infty} \int_0^5 x^{1/2} \cos(nx)\, dx = 0$.

Solution. Method 1. If $x > 0$, then

$$\frac{d}{dx}\left(\frac{x^{1/2}\sin(nx)}{n}\right) = x^{1/2}\cos(nx) + \frac{x^{-1/2}\sin(nx)}{2n}.$$

If $a > 0$, then integrating on the interval $[a, 5]$ and applying the fundamental theorem of calculus shows that

$$\frac{5^{1/2}\sin(5n)}{n} - \frac{a^{1/2}\sin(an)}{n} = \int_a^5 x^{1/2}\cos(nx)\, dx + \frac{1}{2n}\int_a^5 x^{-1/2}\sin(nx)\, dx.$$

(This calculation is equivalent to integrating by parts. The reason for working on the interval $[a, 5]$ instead of on $[0, 5]$ is that $x^{-1/2}$ is unbounded at 0.) The absolute value of the expression on the left-hand side is no greater than $(5^{1/2} + a^{1/2})/n$. Since $|\sin(nx)| \le 1$, the second term on the right-hand side has absolute value bounded above by

$$\frac{1}{2n}\int_a^5 x^{-1/2}\, dx, \qquad \text{or} \qquad \frac{5^{1/2} - a^{1/2}}{n}.$$

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Combining the two inequalities shows that

$$\left|\int_a^5 x^{1/2}\cos(nx)\,dx\right| \le 2\cdot\frac{5^{1/2}}{n}.$$

The expression $x^{1/2}\cos(nx)$ is continuous on the interval $[0,5]$, hence Riemann integrable, so

$$\lim_{a\to 0^+}\int_a^5 x^{1/2}\cos(nx)\,dx = \int_0^5 x^{1/2}\cos(nx)\,dx.$$

Limits preserve the weak order relation, so

$$\left|\int_0^5 x^{1/2}\cos(nx)\,dx\right| \le 2\cdot\frac{5^{1/2}}{n}.$$

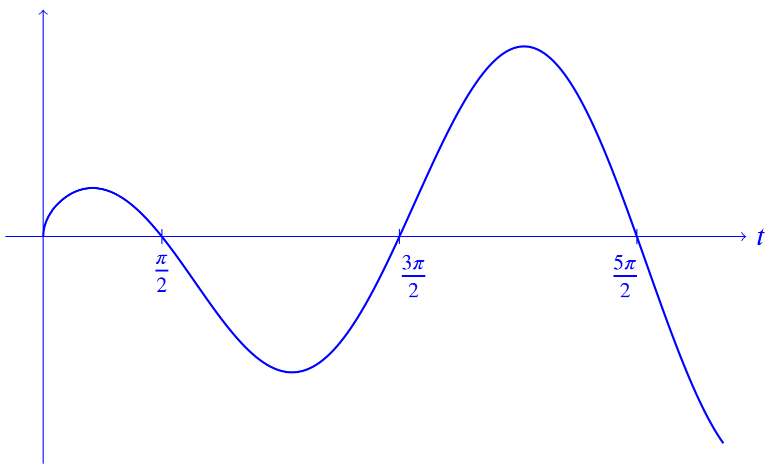
The upper bound evidently tends to 0 when $n\to\infty$, so the squeeze theorem implies that

$$\lim_{n\to\infty}\int_0^5 x^{1/2}\cos(nx)\,dx = 0.$$

Method 2. Introduce a new variable t equal to nx to rewrite the integral as follows:

$$\int_0^5 x^{1/2}\cos(nx)\,dx = \frac{1}{n^{3/2}}\int_0^{5n} t^{1/2}\cos(t)\,dt.$$

Since $t^{1/2}$ is a strictly increasing function of t, the integrand is oscillating with increasing amplitude, as shown in the figure.



The area under the cosine graph from $3\pi/2$ to $5\pi/2$ has the same magnitude as the negatively signed area lying between $\pi/2$ and $3\pi/2$, so the area under the graph of $t^{1/2}\cos(t)$ between $3\pi/2$ and $5\pi/2$ exceeds the magnitude of the negatively signed area for the same graph from $\pi/2$ to $3\pi/2$. Thus

$$\int_{\pi/2}^{5\pi/2} t^{1/2}\cos(t)\,dt > 0.$$

The same reasoning applies for the interval from $5\pi/2$ to $9\pi/2$ and to subsequent intervals of width 2π.

Accordingly, the integral of $t^{1/2}\cos(t)$ from 0 to $5n$ is a sum of some positive integrals, plus the initial integral from 0 to $\pi/2$ (positive too), plus one integral over a partial period ending at $5n$. That last integral could be negative, but the absolute value of the negative part is at most the product of $(5n)^{1/2}$ (an upper bound for the absolute value of the integrand) times π (the maximal width of the final interval where the function is negative). The upshot is that

$$\int_0^{5n} t^{1/2}\cos(t)\,dt > -(5n)^{1/2}\pi.$$

Analogous reasoning shows that the integral of $t^{1/2}\cos(t)$ from $3\pi/2$ to $7\pi/2$ is negative, as are integrals over successive intervals of width 2π. Therefore the whole integral from 0 to $5n$ is a sum of some negative integrals, plus an initial positive integral from 0 to $\pi/2$ (which is less than the product of the width $\pi/2$ and the upper bound $(\pi/2)^{1/2}$ for the integrand), plus a possibly positive final term of size not exceeding $(5n)^{1/2}\pi$. Thus

$$(\pi/2)^{3/2} + (5n)^{1/2}\pi > \int_0^{5n} t^{1/2}\cos(t)\,dt.$$

Putting the two inequalities together shows that

$$\frac{(\pi/2)^{3/2} + (5n)^{1/2}\pi}{n^{3/2}} > \frac{1}{n^{3/2}}\int_0^{5n} t^{1/2}\cos(t)\,dt > \frac{-(5n)^{1/2}\pi}{n^{3/2}}.$$

Both the upper bound and the lower bound have limit equal to 0 when $n \to \infty$, so the squeeze theorem implies that the expression in the middle has limit equal to 0 too.

Remark. This problem is a special case of a general proposition known as the *Riemann–Lebesgue lemma*, according to which every integrable function f on a compact interval $[a,b]$ has the property that

$$\lim_{n\to\infty}\int_a^b f(x)\cos(nx)\,dx = 0 \qquad \text{and} \qquad \lim_{n\to\infty}\int_a^b f(x)\sin(nx)\,dx = 0.$$

The intuitive idea is that when n is large, both $\cos(nx)$ and $\sin(nx)$ oscillate very rapidly, but f changes only a little on a small scale. Therefore the positive contributions to the integral and the negative contributions should approximately cancel.

5. a) State the completeness axiom for the real numbers.

   Solution. See §1.6 of the textbook.

   b) Define the concept of continuity of a function at a point.

   Solution. Several equivalent definitions are stated in §§5.4.2–5.4.3 of the textbook.

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6. Show that $\lim_{x\to 0} \dfrac{(\sin x)^5 - x^5}{(1-\cos x)\sin(x^5)} = -\dfrac{5}{3}$.

Solution. A brute-force application of l'Hôpital's rule is contraindicated, for you would have to differentiate seven times.

Method 1. Lagrange's theorem for Taylor polynomials implies that

$$\sin x = x - \frac{1}{3!}x^3 + O(x^5),$$

where $O(x^5)$ stands for a remainder whose absolute value is bounded above by a constant times $|x|^5$. The constant could be taken to be $1/5!$, but the value of this constant is not important here. Invoking the binomial expansion shows that

$$(\sin x)^5 = x^5 + 5x^4\left(-\frac{1}{3!}x^3\right) + O(x^9).$$

Substituting x^5 for x shows that $\sin(x^5) = x^5 + O(x^{15})$. Also $1 - \cos x = \frac{1}{2}x^2 + O(x^4)$, so

$$\frac{(\sin x)^5 - x^5}{(1-\cos x)\sin(x^5)} = \frac{-\frac{5}{6}x^7 + O(x^9)}{\frac{1}{2}x^7 + O(x^9)} = \frac{-\frac{5}{6} + O(x^2)}{\frac{1}{2} + O(x^2)}.$$

Therefore the limit when $x \to 0$ is equal to $-5/3$.

Method 2. Some trickery makes l'Hôpital's rule a feasible method. Two easy applications of l'Hôpital's rule show that

$$\lim_{x\to 0}\frac{x^2}{1-\cos x} = 2, \qquad \text{so} \qquad \lim_{x\to 0}\frac{(\sin x)^5 - x^5}{(1-\cos x)\sin(x^5)} = 2\lim_{x\to 0}\frac{(\sin x)^5 - x^5}{x^2\sin(x^5)}.$$

Also

$$\lim_{x\to 0}\frac{x}{\sin x} = 1, \qquad \text{so} \qquad \lim_{x\to 0}\frac{x^5}{\sin(x^5)} = 1.$$

Therefore the problem reduces to computing

$$2\lim_{x\to 0}\frac{(\sin x)^5 - x^5}{x^7}.$$

Apply the mean-value theorem to the fifth-power function to see that $u^5 - x^5 = 5c^4(u-x)$ for some c between u and x. Setting u equal to $\sin x$ reveals that $(\sin x)^5 - x^5 = 5c^4(\sin x - x)$ for some c between $\sin x$ and x. Since $(\sin x)/x$ and x/x both approach 1 when $x \to 0$, the squeeze theorem implies that $c/x \to 1$ too. Accordingly, the problem reduces to finding

$$2\lim_{x\to 0}\frac{5x^4(\sin x - x)}{x^7}, \qquad \text{or} \qquad 2\lim_{x\to 0}\frac{5(\sin x - x)}{x^3}.$$

Three easy applications of l'Hôpital's rule now yield the required answer of $-5/3$.

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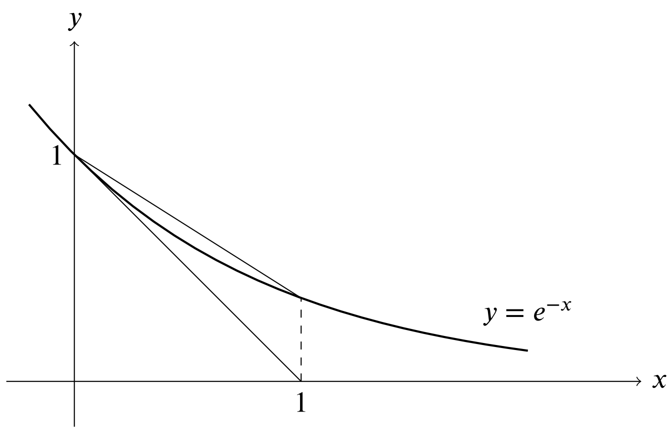
Extra Credit. The number $\Gamma(1/5)$, also known as $5 \cdot (\frac{1}{5}!)$, can be expressed as the convergent improper integral

$$\int_0^\infty x^{-4/5} e^{-x}\,dx.$$

Show that the numerical value of $\Gamma(1/5)$ lies between 4 and 5.
Hint: Consult the captured secret plans below.



Solution. The integrand is positive, so the value of the integral exceeds $\int_0^1 x^{-4/5} e^{-x}\,dx$. The function e^{-x} is convex with slope -1 at the origin, so $e^{-x} \ge 1 - x$, with equality only when $x = 1$. Therefore

$$\int_0^1 x^{-4/5} e^{-x}\,dx > \int_0^1 x^{-4/5}(1-x)\,dx = \left[5x^{1/5} - \frac{5}{6}x^{6/5}\right]_0^1 = 5 - \frac{5}{6} > 4.$$

Thus 4 is a lower bound for the value of the whole integral.

To find an upper bound, observe first that

$$\int_1^\infty x^{-4/5} e^{-x}\,dx < \int_1^\infty e^{-x}\,dx = \lim_{N\to\infty} \int_1^N e^{-x}\,dx = \lim_{N\to\infty} \left[-e^{-N} + e^{-1}\right] = \frac{1}{e}.$$

Convexity implies that when $0 < x < 1$, the graph of e^{-x} lies below the line joining the points $(0,1)$ and $(1, e^{-1})$. The slope of this line is $-(1 - e^{-1})$, so $e^{-x} < 1 - (1 - e^{-1})x$ when $0 < x < 1$. Therefore

$$\int_0^1 x^{-4/5} e^{-x}\,dx < \int_0^1 x^{-4/5} - (1 - e^{-1})x^{1/5}\,dx = \left[5x^{1/5} - \frac{5}{6}(1 - e^{-1})x^{6/5}\right]_0^1 = 5 - \frac{5}{6}(1 - e^{-1}).$$

Adding the upper bounds for the two parts of the integral gives a total upper bound of

$$5 - \frac{5e - 11}{6e}.$$

The value of $5e - 11$ is positive, since $e > 2.2$, so this upper bound is less than 5.

With more work, you could improve on this "back of the envelope" estimation. Asking a calculator or a computer for a precise approximation of the integral reveals that $\Gamma(1/5) \approx 4.59$.