Final Examination

Part A: Sentence Completion

Your answer to each of problems 1–3 should be a complete sentence that starts as indicated.

1. The least-upper-bound property (or completeness property) of the real numbers says that if S is a non-empty set, and S is bounded above, then ....
2. The mean-value theorem states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists ....

   [Warning: Do not confuse the mean-value theorem with the intermediate-value theorem!]
3. A function $f : (a, b) \to \mathbb{R}$ is said to be uniformly continuous if for every positive ε ....

Part B: Examples

Your task in problems 4–5 is to exhibit a concrete example satisfying the indicated property. You should provide a brief explanation of why your example works.

4. Give an example of an integrable function $f : [-1, 1] \to \mathbb{R}$ that is not differentiable at 0.
5. Give an example of a sequence $\{x_k\}_{k=1}^{\infty}$ such that

$$\limsup_{n\to\infty} \sum_{k=1}^{n} x_k \qquad \text{and} \qquad \liminf_{n\to\infty} \sum_{k=1}^{n} x_k$$

   are finite and unequal. In other words, give an example of a divergent series that has bounded partial sums.

Part Γ: Proof

Your proofs should be written in complete sentences, each step being justified. You may invoke theorems from the course if you indicate what the cited theorems say.

6. Suppose $x_n = \cos(e^n)$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.
7. Suppose a function f is differentiable at every point of the interval $(0, 1)$. A standard proposition states that f must then be continuous at every point of the interval $(0, 1)$. Prove this proposition.

Part Δ: Optional Extra Credit Problem

A sequence $\{f_n\}_{n=1}^{\infty}$ of functions with common domain S is said to *converge pointwise* to a limit function f when $\lim_{n\to\infty} f_n(x)$ exists for each x in S and equals $f(x)$. Taking the domain S to be the closed interval $[0, 1]$, give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions that converges pointwise to a discontinuous limit function f.