Exam statistics

Average 88, bravo!

Solutions are posted.

Does every divergent sequence have a convergent subsequence?

No.

Example

If $x_n = n$, then the sequence $\{x_n\}_{n=1}^{\infty}$ increases without bound, and so does every subsequence.

Bolzano–Weierstrass theorem

Bernard Bolzano
(1781–1848)

Karl Weierstrass
(1815–1897)

Theorem

Every **bounded** *sequence of real numbers has a convergent subsequence.*

Proof (different from the one in the textbook)

The B–W theorem is a consequence of the following.

Lemma
Every sequence of real numbers has a monotonic subsequence.

Proof.
If the sequence has no maximum, then there is an increasing subsequence.
Similarly, if some tail of the sequence has no maximum, then there is an increasing subsequence.
Suppose, then, that every tail has a maximum. Let x_{n_1} be the maximum term of the whole sequence. The tail consisting of all terms beyond x_{n_1} has a maximum term, say x_{n_2}. The tail of all terms beyond x_{n_2} has a maximum x_{n_3}, and so on.
The subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ is decreasing. $\square$

Example

What convergent subsequences does $\{\cos\left(\frac{n\pi}{2}\right)\}_{n=1}^{\infty}$ have?
For the subsequence corresponding to odd values of n, we get the subsequence that is constantly equal to 0.
The subsequence corresponding to n equal to a multiple of 4 is constantly equal to 1.
The subsequence corresponding to n being a multiple of 2 that is not a multiple of 4 has the constant value -1.

Definition

The largest limit of any convergent subsequence of a bounded sequence $\{x_n\}_{n=1}^{\infty}$ is the *limit superior*, written $\limsup\limits_{n\to\infty} x_n$.
The smallest limit of any convergent subsequence is the *limit inferior*, written $\liminf\limits_{n\to\infty} x_n$.

Assignment due next class

1. Read subsection 2.3.1 (about upper and lower limits).
2. Write a solution to Exercise 2.3.5.
3. Suppose a sequence $\{x_n\}_{n=1}^{\infty}$ converges to a limit L. Prove that if you reorder the terms of the sequence arbitrarily (for instance, $x_{409}, x_5, x_{17}, x_{270}, x_1, x_{29}, x_{323}, \ldots$), the new sequence still converges, and to the same limit L.