Three views of lim sup of a bounded sequence $\{x_n\}_{n=1}^{\infty}$

1. $\limsup_{n\to\infty} x_n$ is the largest limit of any convergent subsequence of the sequence $\{x_n\}_{n=1}^{\infty}$.
2. $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup\{x_k : k \geq n\}\right) = \inf_n \left(\sup\{x_k : k \geq n\}\right)$
3. $\limsup_{n\to\infty} x_n = L$ means that for every positive number ε,
   - there exists an M such that $x_n < L + \varepsilon$ when $n \geq M$, and also
   - for every M there exists an n greater than M for which $x_n > L - \varepsilon$.

A general question

The real numbers have

1. an algebraic structure
   (addition and multiplication)
2. an order structure
   (the relation $\leq$ makes $\mathbb{R}$ into an ordered field)
3. a metric structure
   ($|x - y|$ represents the distance between x and y)

Is the operation of taking $\limsup$ compatible with these three structures?

lim sup and algebraic operations

Are the following properties true?

$$\limsup_{n\to\infty}(x_n + y_n) \stackrel{?}{=} \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

$$\limsup_{n\to\infty}(x_n y_n) \stackrel{?}{=} \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right)$$

No, because the two sequences might be "out of phase": consider $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

Assignment due next class

1. Read subsections 2.3.2–2.3.3.
2. Prove that
   (a) $\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$
   (b) $\left|\limsup_{n\to\infty} x_n\right| \leq \limsup_{n\to\infty} |x_n|$