Recap: Rolle's Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) = f(b)$, and the derivative f' exists at all points of (a, b), then there is some point c (at least one) in the interval (a, b) for which $f'(c) = 0$.

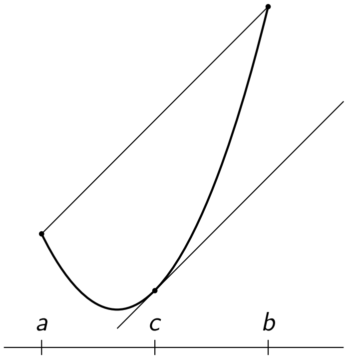
Proof sketch: The derivative must equal 0 at a point of (a, b) where f has a maximum or a minimum.

The "theorem of the mean"

Theorem (mean-value theorem, basic version)

If $f\colon [a,b] \to \mathbb{R}$ *is continuous, and the derivative* f' *exists at all points of* (a,b), *then there exists a point* c *in* (a,b) *for which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Cauchy's form of the mean-value theorem

Theorem (not in the book)

If f and g are two continuous functions on $[a, b]$ that are differentiable at all points of (a, b), then there is a point c in (a, b) for which $g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$.

Proof of the mean-value theorem

Consider the function $g(x)(f(b) - f(a)) - f(x)(g(b) - g(a))$. This continuous function has the same value $f(b)g(a) - g(b)f(a)$ at both endpoints, so by Rolle's theorem, there is a point c where the derivative equals 0. That conclusion proves Cauchy's mean-value theorem.

For the basic version of the mean-value theorem, take g to be the identity function: $g(x) = x$.

Some applications of the mean-value theorem

- If f' is identically equal to 0 on an interval, then f is constant on the interval. [Proposition 4.2.5]
- If $f'(x) > 0$ an interval, then $f(x)$ is strictly increasing. [See Example 4.2.7.]
- If the derivative $f'(x)$ is bounded on an interval, then $f(x)$ is uniformly continuous. [See Exercise 4.2.3.]

Assignment due next class

Write solutions to Exercises 4.2.3 and 4.2.5.