Announcements/reminders

- Class meets Monday (April 30) and Tuesday (May 1).
- I will hold my usual office hour 3:00–4:00 in the afternoon on Tuesday (May 1) and Thursday (May 3).
- The comprehensive final examination takes place 8:00–10:00 on Monday morning (May 7).
- Material for the final exam: sections 0.3, 1.1–1.4, 2.1–2.5, 2.6.1, 2.6.2, 3.1–3.4, 4.1–4.4, 5.1–5.3.

Recap on integrable functions

For a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and $M_k := \sup\{f(x) : x_{k-1} \leq x \leq x_k\}$, the corresponding *upper sum* of f on the interval $[a, b]$ is $\displaystyle\sum_{k=1}^{n} M_k(x_k - x_{k-1})$.

Similarly define lower sums by replacing sup with inf.

If the infimum of all the upper sums equals the supremum of all the lower sums, then f is an *integrable function*.

Comparison between upper sums and lower sums

If extra division points are added to a partition, the upper sum decreases (or stays the same), and the lower sum increases (or stays the same).

If P_1 and P_2 are two partitions, and P_3 contains all the division points of both P_1 and P_2, then

$$\text{lower sum for } P_1 \leq \text{lower sum for } P_3 \leq \text{upper sum for } P_3 \leq \text{upper sum for } P_2$$

Therefore all the lower sums for different partitions are $\leq$ all the upper sums for different partitions.

Proof that continuous functions are integrable

Suppose f is continuous on $[a, b]$. Fix a positive ε.

Since f is automatically uniformly continuous on $[a, b]$, there is a positive δ such that $|f(x) - f(y)| \leq \frac{\varepsilon}{b-a}$ when $|x - y| \leq \delta$.

Choose a natural number n for which $\frac{b-a}{n} < \delta$, and partition $[a, b]$ into n subintervals of equal width $\frac{b-a}{n}$. On each subinterval, the difference between $\sup f(x)$ and $\inf f(x)$ is at most $\frac{\varepsilon}{b-a}$.

Therefore the upper sum for this partition and the lower sum differ by no more than $n \cdot \frac{\varepsilon}{b-a} \cdot \frac{b-a}{n}$, or ε.

Consequently, the infimum of all upper sums and the supremum of all lower sums differ by at most ε. But ε is arbitrary, so f is an integrable function.

Practice exercise (not to hand in)

Suppose $f\colon [0,1] \to \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} 1/2, & \text{if } 1/2 < x \leq 1, \\ 1/4, & \text{if } 1/4 < x \leq 1/2, \\ & \dots \\ 1/2^n, & \text{if } 1/2^n < x \leq 1/2^{n-1} \\ & \dots \\ 0, & \text{if } x = 0. \end{cases}$$

Show that f is integrable (even though f has infinitely many points of discontinuity!), and $\int_0^1 f = 1/3$.