Examination 1

Part A: Sentence Completion

Your answer to each of problems 1–3 should be a complete sentence that starts as indicated.

1. The statement "$\lim_{n\to\infty} x_n = L$" means that for every positive real number ε, there exists ....

   Solution. The statement "$\lim_{n\to\infty} x_n = L$" means that for every positive number ε, there exists a number M such that $|x_n - L| < \varepsilon$ when $n \geq M$. [Definition 2.1.2]

2. The set of real numbers is the only ordered field that additionally ....

   Solution. The set of real numbers is the only ordered field that additionally has the least-upper-bound property (the completeness property). [Theorem 1.2.1]

3. The Archimedean property states that ....

   Solution. The Archimedean property states that if x is a positive real number, and y is a real number, then there exists a natural number n such that $nx > y$. [Theorem 1.2.4]

   A simpler, equivalent version of the Archimedean property is that the set of natural numbers has no upper bound in the real numbers.

Part B: Examples

Your task in problems 4–5 is to exhibit a concrete example satisfying the indicated property. You should provide a brief explanation of why your example works.

4. Give an example of a set of real numbers that has a supremum but not a maximum.

   Solution. One example is the open interval $(0, 1)$. The supremum (least upper bound) is the number 1, but this number is not an element of the interval, so the number 1 is not a maximum.

5. Give an example of a bounded sequence of real numbers that does not converge.

   Solution. One example is the sequence $\{(-1)^n\}_{n=1}^{\infty}$, the sequence of alternating -1 and 1. The sequence is bounded, since all the terms have absolute value equal to 1. The sequence does not converge, for every two consecutive terms have distance 2 from each other.

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Part Γ: Proof

Your proof should be written in complete sentences, each step being justified. You may invoke theorems from the textbook, in which case you should indicate what the cited theorems say.

6. Suppose $x_1 = 5$, and $x_{n+1} = \dfrac{1+x_n}{2}$ when $n \geq 1$. Prove that this recursively defined sequence $\{x_n\}_{n=1}^{\infty}$ converges.

Solution. Method 1. Apply the monotone convergence theorem, one version of which says that if a sequence is decreasing and bounded below, then the sequence must converge (and the limit is the infimum of the terms of the sequence).

To show by induction that the sequence is bounded below by 0, first observe that $x_1 = 5 > 0$, so the basis step holds. Next, if n is a natural number for which $x_n > 0$, then $x_{n+1} = \frac{1+x_n}{2} > \frac{1+0}{2} > 0$. Accordingly, the induction step holds. By mathematical induction, all the terms of the sequence are greater than 0.

To show by induction that the sequence is decreasing, first observe that

$$x_2 = \frac{1+x_1}{2} = \frac{1+5}{2} = 3 < 5 = x_1,$$

so $x_2 < x_1$ (the basis step). Next, if n is a natural number for which $x_{n+1} < x_n$, then

$$x_{n+2} = \frac{1+x_{n+1}}{2} < \frac{1+x_n}{2} = x_{n+1}, \qquad \text{so } x_{n+2} < x_{n+1}.$$

Accordingly, the induction step holds. By mathematical induction, the sequence is (strictly) decreasing.

Since the sequence is decreasing and bounded below, the monotone convergence theorem implies that the sequence converges to some limit L.

Remark. The problem does not ask for the value of L, but this value is easy to find. Passing to the limit in the equation defining the sequence shows that

$$L = \frac{1+L}{2}, \qquad \text{so} \qquad L = 1.$$

Method 2. In view of the preceding remark, the only candidate for the limit is the number 1, so let a_n denote the quantity $x_n - 1$. Showing that $\lim_{n\to\infty} x_n$ exists and equals 1 is equivalent to showing that $\lim_{n\to\infty} a_n$ exists and equals 0. If n is an arbitrary natural number, then

$$a_{n+1} = x_{n+1} - 1 = \frac{1+x_n}{2} - 1 = \frac{x_n - 1}{2} = \frac{1}{2}a_n.$$

Thus the sequence $\{a_n\}_{n=1}^{\infty}$ is a geometric sequence with multiplier $\frac{1}{2}$. A proposition about geometric sequences [Proposition 2.2.11] states that a geometric sequence with positive multiplier less than 1 converges to 0. So $\lim_{n\to\infty} a_n = 0$, as required.

Method 3. It is possible to solve the recursion to find a closed-form expression for the general term of this particular sequence. The pattern is that each term equals $\frac{1}{2}$ plus $\frac{1}{2}$ times the preceding term, so by writing out the first few terms you may be able to guess that $x_n = 1 + \frac{8}{2^n}$. Once you have guessed this formula, you can prove the validity of the formula by induction. For the basis step ($n = 1$), observe that $1 + \frac{8}{2^1} = 5 = x_1$. For the induction step, suppose it is known for a certain natural number n that $x_n = 1 + \frac{8}{2^n}$. Then

$$x_{n+1} = \frac{1 + x_n}{2} = \frac{1 + 1 + \frac{8}{2^n}}{2} = 1 + \frac{8}{2^{n+1}},$$

so the indicated formula holds for the next natural number. By induction, the proposed formula for x_n holds for every natural number n.

With an explicit formula for the general term in hand, you can apply standard theorems about limits. The sequence $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ is a known geometric sequence that has limit 0. Since limits preserve products, $\lim_{n\to\infty} \frac{8}{2^n} = 0$. Since limits preserve sums, $\lim_{n\to\infty} \left(1 + \frac{8}{2^n}\right) = 1$. Thus $\lim_{n\to\infty} x_n$ exists and equals 1.

Part Δ: Optional Extra Credit Problem

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose

$$y_n = x_n^2 \qquad \text{and} \qquad z_n = \frac{x_n}{x_n^2 + x_n + 1} \qquad \text{when } n \in \mathbb{N}.$$

Prove that if both of the sequences $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge, then the sequence $\{x_n\}_{n=1}^{\infty}$ must converge too.

Solution. A useful initial observation is that the expression for z_n is well defined: the denominator is never equal to 0. Indeed, completing the square shows that the quadratic expression $x^2 + x + 1$ can be rewritten as $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$, a quantity that is positive for every real number x (squares cannot be negative in an ordered field).

Method 1. Solve for x_n in terms of y_n and z_n. Multiplying by the denominator in the expression for z_n shows that

$$\begin{aligned} z_n(x_n^2 + x_n + 1) &= x_n, && \text{or} \\ z_n y_n + z_n &= x_n(1 - z_n), && \text{or} \\ \frac{z_n y_n + z_n}{1 - z_n} &= x_n && \text{as long as } z_n \neq 1. \end{aligned}$$

Limits respect sums, products, and quotients, as long as there is no division by 0, so the hypothesis of existence of $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} z_n$ implies that $\lim_{n\to\infty} x_n$ exists too, as long as $\lim_{n\to\infty} z_n \neq 1$.

Here is one way to verify that $\lim_{n\to\infty} z_n \neq 1$. If x is an arbitrary real number, then

$$0 \le (x-1)^2 = x^2 - 2x + 1, \qquad \text{so} \qquad 2x \le x^2 + 1.$$

Adding x to both sides shows that $3x \le x^2 + x + 1$. As previously observed, the expression $x^2 + x + 1$ is positive, so the inequality is preserved by dividing both sides by $3(x^2 + x + 1)$: namely,

$$\frac{x}{x^2 + x + 1} \le \frac{1}{3}.$$

Replacing x by x_n reveals that $z_n \le 1/3$ for every natural number n, so certainly $z_n \neq 1$. Moreover, limits preserve the weak order relation, so $\lim_{n\to\infty} z_n \le 1/3$; in particular, $\lim_{n\to\infty} z_n \neq 1$. This deduction completes the proof.

Method 2. There is a theorem [Proposition 2.2.6] stating that if a sequence $\{a_n\}_{n=1}^{\infty}$ converges, and if $a_n \ge 0$ for every n, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\lim_{n\to\infty} a_n}$. Applying this statement to the convergent sequence $\{y_n\}_{n=1}^{\infty}$ shows that the sequence $\{|x_n|\}_{n=1}^{\infty}$ converges.

Now consider three cases. If there exists some natural number M such that $x_n \ge 0$ whenever $n \ge M$, then the tail sequences $\{|x_n|\}_{n=M}^{\infty}$ and $\{x_n\}_{n=M}^{\infty}$ are identical. But the M-tail of a sequence converges if and only if the whole sequence converges, so convergence of the sequence $\{|x_n|\}_{n=1}^{\infty}$ implies convergence of the sequence $\{x_n\}_{n=1}^{\infty}$.

Similarly, if there exists some natural number M such that $x_n \le 0$ whenever $n \ge M$, then the same argument shows that the sequence $\{-x_n\}_{n=1}^{\infty}$ converges. The product theorem for limits implies that the original sequence $\{x_n\}_{n=1}^{\infty}$ converges.

The third case is that neither of the two preceding situations holds. Negating the hypotheses of the first two cases shows that for every natural number M, there is some value of n greater than or equal to M for which $x_n < 0$, and there is some other value of n greater than or equal to M for which $x_n > 0$. In other words, there is one subsequence of $\{x_n\}_{n=1}^{\infty}$ consisting of negative numbers, and there is a second subsequence of $\{x_n\}_{n=1}^{\infty}$ consisting of positive numbers. Since the denominator of z_n is always positive, and the numerator of z_n equals x_n, the numbers z_n and x_n have the same sign for every natural number n. Accordingly, there is one subsequence of $\{z_n\}_{n=1}^{\infty}$ consisting of negative numbers, and there is a second subsequence of $\{z_n\}_{n=1}^{\infty}$ consisting

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of positive numbers. But the sequence $\{z_n\}_{n=1}^\infty$ converges by hypothesis, so every subsequence of $\{z_n\}_{n=1}^\infty$ must have the same limit. Since taking limits preserves the weak order relation, the value of $\lim_{n\to\infty} z_n$ is simultaneously ≤ 0 and ≥ 0, hence equal to 0.

What remains to do in the third case is to leverage the knowledge that $\lim_{n\to\infty} z_n = 0$ to deduce that $\lim_{n\to\infty} x_n$ exists and equals 0. Convergent sequences are in particular bounded [Proposition 2.1.7], so the hypothesis of convergence of the sequence $\{y_n\}_{n=1}^\infty$ yields the existence of a positive number B such that $x_n^2 \le B$, hence $|x_n| \le \sqrt{B}$, for every natural number n. The triangle inequality implies that

$$|x_n| = \left|z_n\left(x_n^2 + x_n + 1\right)\right| \le |z_n|\left(B + \sqrt{B} + 1\right) \qquad \text{for every } n.$$

The squeeze theorem (or the comparison theorem) now implies that $\lim_{n\to\infty} x_n = 0$.