Advanced Calculus I

Instructions   Please write your solutions on your own paper.

These problems should be treated as essay questions. A problem that says "give an example" requires a supporting explanation. In all problems, you should explain your reasoning in complete sentences.

Students in Section 501 should answer questions 1–6 in Parts A and B, and optionally the extra-credit question 10 in Section D.

Students in Section 200 (the honors section) should answer questions 1–3 in Part A and questions 7–9 in Part C, and optionally the extra-credit question 10 in Section D.

Part A, for both Section 200 and Section 501

In this part of the exam, your task is to analyze the following proposition from three different points of view.

**Proposition.** *If n is a natural number, then $e^n > 1 + n$.*

1. Prove the proposition by induction on n. [Recall that $e \approx 2.718$.]

   Solution. Evidently $e^1 > 1 + 1$, since $e \approx 2.718$, so the basis step of the induction holds.

   Suppose it has been shown that $e^n > 1 + n$ for a certain natural number n. Multiplying both sides by e shows that $e^{n+1} > e + en > 2 + en > 2 + n = 1 + (n+1)$. Thus the validity of the proposition for a certain natural number implies the validity of the proposition for the next natural number.

   Since the basis step holds, and the induction step holds, the proposition is valid for every natural number.

2. The figure below suggests the more general statement that $e^x > 1 + x$ for every nonzero real number x (because the graph of e^x is convex).
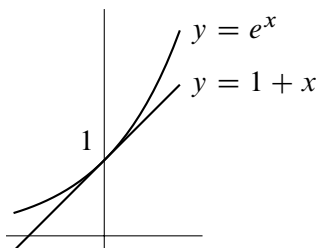
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Prove that $e^x > 1+x$ for every nonzero real number x by writing the Taylor polynomial for e^x of degree 1 and examining the remainder term.

[Recall Lagrange's theorem about approximation by Taylor polynomials: $f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-x_0)^{n+1}$ for some point c between x_0 and x. Take x_0 equal to 0 and n equal to 1.]

Solution. If $f(x) = e^x$, then every derivative of $f(x)$ equals e^x, so $f(0) = 1$ and $f'(0) = 1$. Then Taylor's formula says that $e^x = 1 + x + e^c x^2/2$ for some point c between 0 and x. Since $x^2 > 0$ when $x \neq 0$, and $e^c > 0$ for every point c, the remainder term is positive. Thus $e^x > 1 + x$ when $x \neq 0$.

3. Evidently $e^x > 1$ for every positive real number x. By integrating on a suitable interval, deduce that $e^x > 1 + x$ for every positive real number x.

Solution. The continuous function $e^t - 1$ is positive when $t > 0$, so if $x > 0$, then

$$0 < \int_0^x (e^t - 1)\, dt = \left[e^t - t \right]_0^x = (e^x - x) - (e^0 - 0) = e^x - x - 1.$$

Adding $1 + x$ to both sides shows that $1 + x < e^x$ when $x > 0$, as claimed.

Part B, for Section 501 only

4. State the following theorems:

    (a) the Bolzano–Weierstrass theorem about sequences;

    (b) the squeeze theorem (sandwich theorem) about limits of functions;

    (c) Rolle's theorem about differentiable functions.

Solution. The statements are in the textbook (Theorems 2.40, 5.21, and 7.19).

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5. Give an example of a function f that satisfies all three of the following properties: the function f is continuous on the open interval $(0, 2)$, the function f is not differentiable at the point 1, and $\int_0^2 f(t)\,dt$ is a divergent improper integral.

   Solution. One way to make the function continuous but nondifferentiable at the point 1 is to use a piecewise definition like this:

   $$f(x) = \begin{cases} 1/x, & \text{if } 0 < x < 1, \\ 1, & \text{if } 1 \leq x. \end{cases}$$

   The same effect can be achieved without explicitly introducing cases: namely,

   $$f(x) = \max(1/x, 1).$$

   A different example is $|x - 1| + \frac{1}{x}$. You know from class that the improper integral $\int_0^1 \frac{1}{x}\,dx$ diverges.

   There are many other possible examples.

6. Prove that $\displaystyle\lim_{n\to\infty} \frac{e^{\cos(n)}}{n} = 0$.

   Solution. Evidently $0 < e^{\cos(n)} < 3$ for every n, since the cosine function never exceeds 1 (and $e < 3$). Therefore

   $$0 < \frac{e^{\cos(n)}}{n} < \frac{3}{n}.$$

   But any constant times $1/n$ has limit 0 when $n \to \infty$. The squeeze theorem implies that $e^{\cos(n)}/n$ also has limit 0. Alternatively, you could go back to the definition of limit: if a positive ε is specified, and $N = \lceil 3/\varepsilon \rceil$, then $0 < e^{\cos(n)}/n < 3/n < \varepsilon$ when $n > N$.

Part C, for Section 200 only

7. State the following theorems:

   (a) the Heine–Borel theorem characterizing compact sets of real numbers;

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(b) some theorem from this course in which the word "countable" appears;

(c) some (other) theorem from this course named after Cantor.

Solution. The Heine–Borel theorem states that a set of real numbers is compact if and only if every covering of the set by open intervals can be reduced to a finite subcover. Some results in the textbook concerning countability are Theorems 4.35, 5.60, and 5.63 and Corollary 8.19. One of Cantor's theorems is the uncountability of the real numbers (Theorem 2.4); another is the theorem on nested closed sets (Corollary 4.25).

8. Give an example of a function f that satisfies all of the following properties: the function f is continuous on the open interval $(0, 1)$, the upper limit $\limsup_{x \to 0+} f(x)$ equals ∞, the lower limit $\liminf_{x \to 0+} f(x)$ equals 0, and the integral $\int_0^1 f(t)\, dt$ is a convergent improper integral.

Solution. Since $\int_0^1 1/\sqrt{x}\, dx$ converges, so does an integral with a smaller positive integrand. Here is one example that fits the specified conditions:

$$f(x) = \frac{\sin^2(1/x)}{\sqrt{x}}.$$

There are many other possible examples.

9. Prove that $\displaystyle \lim_{n \to \infty} \int_n^{n+1} \frac{t}{1 + t^3}\, dt = 0$.

Solution. In principle, the integral can be evaluated explicitly, but you do not want to carry out that computation. You can, however, make the following estimate:

$$0 < \int_n^{n+1} \frac{t}{1 + t^3}\, dt < \int_n^{n+1} \frac{t}{t^3}\, dt = \int_n^{n+1} \frac{1}{t^2}\, dt = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n}.$$

The squeeze theorem now implies that the limit of the integral equals 0.

Alternatively, observe that $\int_1^{\infty} \frac{t}{1+t^3}\, dt$ is a convergent improper integral (the integrand is less than $1/t^2$, and you know from class that $\int_1^{\infty} 1/t^2\, dt$ converges). The integrals $\int_1^n \frac{t}{1+t^3}\, dt$ therefore form a bounded increasing sequence, hence a Cauchy sequence. Since the integral $\int_n^{n+1} \frac{t}{1+t^3}\, dt$ is the difference of two consecutive terms of this Cauchy sequence, the integral has limit 0.

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Part D, optional extra-credit question
for both Section 200 and Section 501

10. Alfie, Beth, Gemma, and Delma are studying the limit

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 9x} - \sqrt{x^2 + 4x} \right).$$

Alfie says, "simplifying the square roots converts the problem to

$$\lim_{x \to \infty} \left[(x + 3\sqrt{x}) - (x + 2\sqrt{x}) \right] = \lim_{x \to \infty} \sqrt{x} = \infty."$$

Beth says, "the limit of a sum is the sum of the limits, so the answer is

$$\lim_{x \to \infty} \sqrt{x^2 + 9x} - \lim_{x \to \infty} \sqrt{x^2 + 4x} = \infty - \infty = 0."$$

Gemma says, "by the mean-value theorem, $\sqrt{u} - \sqrt{v} = \frac{1}{2\sqrt{c}}(u - v)$, so the limit equals

$$\lim_{x \to \infty} \frac{1}{2\sqrt{c}} \cdot [(x^2 + 9x) - (x^2 + 4x)] = \lim_{x \to \infty} \frac{1}{2\sqrt{c}} \cdot 5x = \infty."$$

Delma says, "by l'Hôpital's rule, the limit equals

$$\lim_{x \to \infty} \left(\frac{2x + 9}{2\sqrt{x^2 + 9x}} - \frac{2x + 4}{2\sqrt{x^2 + 4x}} \right) = \lim_{x \to \infty} \left(\frac{2x + 9}{2x\sqrt{1 + \frac{9}{x}}} - \frac{2x + 4}{2x\sqrt{1 + \frac{4}{x}}} \right)$$
$$= 1 - 1 = 0."$$

Who (if anyone) is right, and why? Identify the mistakes in the erroneous arguments.

Solution. Everyone is wrong: the value of the limit is actually 5/2.

Alfie's algebra is faulty: the square root of a sum is not equal to the sum of the square roots.

Beth overlooks that the limit law applies only if the limits exist as finite values. The expression $\infty - \infty$ is undefined, an indication that a subtler argument is needed.

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Gemma has a great idea, but the execution falls short. The intermediate point c is not a constant but depends on x. Indeed, the value of c lies between $x^2 + 4x$ and $x^2 + 9x$. Therefore

$$x = \sqrt{x^2} < \sqrt{x^2 + 4x} < \sqrt{c} < \sqrt{x^2 + 9x} = x\sqrt{1 + (9/x)}.$$

Accordingly,

$$\frac{5}{2\sqrt{1 + (9/x)}} < \frac{5x}{2\sqrt{c}} < \frac{5}{2}.$$

Since the limit of the left-hand side as $x \to \infty$ equals $5/2$, the squeeze theorem implies that the original limit equals $5/2$. Thus Gemma's argument can be corrected to give the right answer.

Delma is wrong because l'Hôpital's rule does not apply to undefined expressions of the form $\infty - \infty$. The standard trick for simplifying a difference of square roots algebraically is the following:

$$\left(\sqrt{x^2 + 9x} - \sqrt{x^2 + 4x}\right) \cdot \frac{\sqrt{x^2 + 9x} + \sqrt{x^2 + 4x}}{\sqrt{x^2 + 9x} + \sqrt{x^2 + 4x}}$$

$$= \frac{(x^2 + 9x) - (x^2 + 4x)}{\sqrt{x^2 + 9x} + \sqrt{x^2 + 4x}} = \frac{5x}{\sqrt{x^2 + 9x} + \sqrt{x^2 + 4x}}$$

$$= \frac{5}{\sqrt{1 + (9/x)} + \sqrt{1 + (4/x)}}.$$

Now it is evident that the limit as $x \to \infty$ is equal to $5/2$.