§1	§2	§3	style	total

Math 415 Examination 2 Fall 2006

Modern Algebra I

Please print your **name**: Answer Key

1 True/false

Circle the correct answer; no explanation is required. Each problem in this section counts 5 points.

1. Every group of order n is isomorphic to some subgroup of the symmetric group S_n. `True` `False`

   Solution. True. This is a special case of (the proof of) Cayley's theorem.

2. A subgroup H of a group G is a normal subgroup if and only if the number of left cosets of H is equal to the number of right cosets of H. `True` `False`

   Solution. False. For every subgroup, the *number* of left cosets is equal to the number of right cosets. Normality means that every left coset *is* a right coset.

3. Some abelian group of order 45 has a subgroup of order 10. `True` `False`

   Solution. False. According to Lagrange's theorem, the order of a subgroup divides the order of the group, but 10 does not divide 45.

4. Every abelian group of order 45 has a subgroup of order 9. `True` `False`

   Solution. True. This follows from the fundamental theorem of finite abelian groups, and it is a special case of Theorem 11.16 on page 109 of the textbook.

5. If G is a group and H is a normal subgroup of G, then G is isomorphic to the direct product group $(G/H) \times H$. `True` `False`

Solution. False. For example, suppose $G = \mathbb{Z}_4$, and H is the subgroup $\{0, 2\}$. Then G/H and H are both isomorphic to $\mathbb{Z}_2$, but $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. For another example, suppose $G = S_3$, and H is the cyclic subgroup generated by the 3-cycle $(1, 2, 3)$. Then G/H is isomorphic to $\mathbb{Z}_2$, and H is isomorphic to $\mathbb{Z}_3$, so $(G/H)\times H$ is abelian, but G is not abelian.

2 Short answer

Fill in the blanks; no explanation is required. Each problem in this section counts 5 points.

6. The permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$ generates a cyclic subgroup of the symmetric group S_5. This cyclic subgroup is isomorphic to the direct product group __________ .

   Solution. The given permutation is the product $(2, 5)(1, 3, 4)$ of disjoint cycles. Since the lengths of the cycles are relatively prime, the generated subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.

7. The product $(1, 6, 3)(3, 5, 4)$ of *non-disjoint* cycles represents an element of the symmetric group S_6. The order of this element equals ________ .

   Solution. The product equals the cycle $(1, 6, 3, 5, 4)$, which has order 5.

8. In the direct product group $\mathbb{Z}_{12} \times \mathbb{Z}_{15} \times \mathbb{Z}_{18}$, the element $(9, 10, 11)$ has order ________ .

   Solution. Since $\gcd(9, 12) = 3$, the element 9 has order 4 in $\mathbb{Z}_{12}$. Since $\gcd(10, 15) = 5$, the element 10 has order 3 in $\mathbb{Z}_{15}$. Since $\gcd(11, 18) = 1$, the element 11 has order 18 in $\mathbb{Z}_{18}$. The order of the element $(9, 10, 11)$ in the direct product group equals $\operatorname{lcm}(4, 3, 18)$, or 36.

9. Suppose $\phi\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is a homomorphism such that $\phi(1,0)=12$ and $\phi(0,1)=30$. Then the image of ϕ is a subgroup of $\mathbb{Z}$ whose index equals ________ .

   Solution. The image of ϕ consists of all expressions of the form $12n+30m$ as n and m range over the integers. This is the subgroup of $\mathbb{Z}$ generated by $\gcd(12,30)$: namely, the subgroup $6\mathbb{Z}$. The index of this subgroup (that is, the number of cosets) equals 6.

10. If H is a subgroup of the cyclic group $\mathbb{Z}_{16}$, and the factor group $\mathbb{Z}_{16}/H$ is a simple group, then H is isomorphic to the cyclic group ________ .

    Solution. The subgroup H is a maximal normal subgroup of $\mathbb{Z}_{16}$. Since $\mathbb{Z}_{16}$ is abelian, all subgroups are normal, so H is a maximal proper subgroup. Then $H=\{0,2,4,6,8,10,12,14\}$, and H is isomorphic to $\mathbb{Z}_8$.

3 Essay questions

In the following problems, you must give an explanation. (Continue on the back if you need more space.) Each problem counts 15 points. In addition, this section as a whole carries 5 style points based on how well your solutions are written.

11. Suppose that G is a group, H is a subgroup of G, and K is a subgroup of H; then K is also a subgroup of G. In symbols, $K\le H\le G$. Prove that if additionally K is a *normal* subgroup of G, then K is a *normal* subgroup of H.
    [Warning: the converse statement is false, so be careful that your argument does not overshoot the target.]

    Solution. Since K is a normal subgroup of G, we know that $gK=Kg$ for every element g of G. Now every element of H is, in particular, an element of G, so it follows that $hK=Kh$ for every element h of H. Therefore K is a normal subgroup of H.

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12. Are the direct product groups $\mathbb{Z}_6 \times \mathbb{Z}_6$ and $\mathbb{Z}_4 \times \mathbb{Z}_9$ isomorphic to each other? Explain why or why not.

    Solution. It is implicit in the fundamental theorem of finite abelian groups that these two groups are not isomorphic. One way to see this explicitly is that in $\mathbb{Z}_6 \times \mathbb{Z}_6$, adding any element to itself 6 times produces the identity element, while in $\mathbb{Z}_4 \times \mathbb{Z}_9$, the element $(0, 1)$ does not have this property (in fact, it has order 9).

13. Recall that the alternating group A_4 has 12 elements: namely, the identity permutation; the eight 3-cycles $(1, 2, 3)$, $(1, 3, 2)$, $(1, 2, 4)$, $(1, 4, 2)$, $(1, 3, 4)$, $(1, 4, 3)$, $(2, 3, 4)$, and $(2, 4, 3)$; and the three products of disjoint transpositions $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, and $(1, 4)(2, 3)$.

    Show that if $\phi\colon A_4 \to \mathbb{Z}_4$ is a homomorphism, then ϕ must be the trivial homomorphism that sends every element to the identity element.

    Solution. If σ is any 3-cycle, then σ^3 equals the identity element, and therefore $0 = \phi(\sigma^3) = \phi(\sigma) + \phi(\sigma) + \phi(\sigma)$ in $\mathbb{Z}_4$. But in $\mathbb{Z}_4$, we have $1 + 1 + 1 = 3$ and $2 + 2 + 2 = 2$ and $3 + 3 + 3 = 1$. Consequently, $\phi(\sigma)$ is not 1 or 2 or 3, so $\phi(\sigma) = 0$.

    Thus all eight of the 3-cycles are in the kernel of ϕ. But the kernel of ϕ is a subgroup of A_4, so the order of the kernel is a divisor of 12. Since the kernel has at least eight elements in it, the only way its order can divide 12 is if the kernel is all of A_4. In other words, ϕ sends every element of A_4 to 0.