# Applied Algebra

**Instructions** Please answer these questions on your own paper. Explain your work in complete sentences.

1. Write the permutation (1 2 3)(3 4 5)(4 5 6) as a product of *disjoint* cycles.

2. What is the highest possible order of an element of the symmetric group $S(10)$?

3. Consider the operation $*$ defined on the positive real numbers as follows: $a * b = 5ab$ (where $ab$ on the right-hand side denotes ordinary multiplication). Does this operation $*$ provide the positive real numbers with a group structure?

4. Consider the set of $2 \times 2$ matrices of the form $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$, where $a$ and $b$ are elements of $\mathbb{Z}_2$ (the integers mod 2). Suppose such matrices are added and multiplied in the usual way, but with the arithmetic done in $\mathbb{Z}_2$. Is this structure a ring?

5. Suppose $H$ and $K$ are subgroups of a group $G$. Is the union $H \cup K$ necessarily a subgroup of $G$?

6. Let $G$ denote the multiplicative group of invertible congruence classes of integers modulo 15. The group $G$ has subgroups of what orders?

7. Give an example of two finite groups of the same order that are not isomorphic groups.

8. Suppose a coding function $f\colon \mathbf{B}^3 \to \mathbf{B}^6$ is determined by the generator matrix
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$
Suppose a message encoded by this function is received with errors as
$$101101 \quad 010101 \quad 011111.$$
Decode the received message.
[If you write your decoded message as three words in $\mathbf{B}^3$ and convert each binary word into an equivalent single decimal digit, then you will know if you have the right answer.]

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**Bonus problem for extra credit** Complete the following group table. (Although the group elements are labeled 1 through 9, the group operation $*$ is neither ordinary addition nor ordinary multiplication, and the number 1 is not the identity element.)

| $*$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 9 | | | | | | | | 6 |
| 2 | | 6 | | | | | | 7 | |
| 3 | | | 8 | | | | 5 | | |
| 4 | | | | 7 | | 1 | | | |
| 5 | | | | | 5 | | | | |
| 6 | | | | 1 | | 2 | | | |
| 7 | | | 5 | | | | 1 | | |
| 8 | | 7 | | | | | | 4 | |
| 9 | 6 | | | | | | | | 3 |

Notice that this problem is not a sudoku! On the one hand, there is no constraint on $3 \times 3$ subsquares. On the other hand, you have the full power of a group law to help fill in the entries.