Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Suppose that F is a field in which $(x+y)^2 = x^2 + y^2$ for all elements x and y. Show that F has characteristic 2 (that is, $1+1=0$ in F).

   Solution. Set both x and y equal to the multiplicative identity element 1:
   $$(1+1)^2 = 1^2 + 1^2.$$
   Distributing the multiplication on the left-hand side shows that
   $$1^2 + 1^2 + 1^2 + 1^2 = 1^2 + 1^2.$$
   Cancel the common terms to see that
   $$1^2 + 1^2 = 0.$$
   But $1^2 = 1$, since 1 is the multiplicative identity, so
   $$1 + 1 = 0.$$
   Thus the field has characteristic equal to 2.

   Remark We did a related calculation in class yesterday to prove that every Boolean ring is commutative.

   The condition stated in the problem is actually both necessary and sufficient for the field to have characteristic 2.

2. Consider the set of 2×2 matrices with real entries. Suppose that matrices are added and multiplied in the usual way, but a nonstandard scalar multiplication rule is defined as follows:
   $$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & b \\ \lambda c & d \end{pmatrix} \qquad \text{for every real number } \lambda.$$
   Do these operations provide the set of 2×2 matrices with the structure of an algebra (a ring that is simultaneously a vector space)? Explain.

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Solution. The answer is negative, for the indicated scalar multiplication does not provide a correct vector-space structure. The property that fails is the property stating that $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$ for every vector-space element v and all field elements λ_1 and λ_2. Indeed,

$$(1+1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & b \\ 2c & d \end{pmatrix}$$

according to the nonstandard rule for scalar multiplication, but

$$1\begin{pmatrix} a & b \\ c & d \end{pmatrix} + 1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix};$$

the two expressions do not match if either $b \neq 0$ or $d \neq 0$.

Remark This problem is similar to Exercise 9 on page 197 in section 4.4.