Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Suppose H is a subgroup of a group G. If a is an element of G such that the left coset aH equals the right coset Ha, must the element a commute with every element of H?

   Solution. All that can be deduced is the following: to each element h_1 of H there corresponds an element h_2 of H such that $ah_1 = h_2a$ (and conversely). But the two elements h_1 and h_2 need not be equal.

   An explicit example can be found in the simplest noncommutative group that you know: the symmetric group $S(3)$. Let H be the subgroup $\{id, (1\ 2\ 3), (1\ 3\ 2)\}$, and take a to be the transposition $(1\ 2)$. Then the left coset $(1\ 2)H$ and the right coset $H(1\ 2)$ both equal $\{(1\ 2), (1\ 3), (2\ 3)\}$. (You do not even need to calculate to find this coset: since $S(3)$ has order 6, the only coset—either left or right—besides H itself is $\{(1\ 2), (1\ 3), (2\ 3)\}$.) But the transposition $(1\ 2)$ does not commute with the 3-cycle $(1\ 2\ 3)$: indeed, $(1\ 2)(1\ 2\ 3) = (2\ 3)$ and $(1\ 2\ 3)(1\ 2) = (1\ 3)$.

2. Consider the coding function that sends each two-character word ab to the codeword $abaabb$. How many errors will this code correct?

   Solution. Method 1. You can see without using any theory that this code corrects one error. The codewords have the property that the first, third, and fourth letters match, and also the second, fifth, and sixth letters match. In a word with one error, one of these sets of three letters will have a letter that does not match, and the error can be corrected by "majority rule". (On the other hand, if two letters in a set of three are wrong, then majority rule will fail: the code does not correct two errors.)

   Notice that composing this coding function with a permutation that switches the fourth and fifth letters gives the "tell me three times" code, which you know (from class or from page 233 in the textbook) corrects one error.

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Method 2. You can list all the codewords (assuming that the code is a binary code): namely, 000000, 101100, 010011, 111111. All the codewords differ from each other in at least 3 positions (in the terminology that we used in class today, the minimum weight of a nonzero codeword is 3), so the code corrects 1 error (in Theorem 5.4.2 on page 236 in the textbook, $k = 1$).