Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Prove by induction that $\dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$ for every positive integer n.

   Solution. When $n = 1$, the equation reduces to the true statement $\frac{1}{2} = 1 - \frac{1}{2}$, so the basis step holds.

   Now suppose that the equality
   $$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$$
   is known for a certain positive integer k. Adding $\frac{1}{2^{k+1}}$ to both sides shows that
   $$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}},$$
   and since $\frac{1}{2^k} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}}$, the right-hand side reduces to $1 - \frac{1}{2^{k+1}}$. Therefore the statement for integer $k+1$ is a consequence of the statement for integer k.

   By induction, the statement holds for every positive integer n.

   Remark This problem is a special case of Exercise 7 on page 24 of the textbook, a problem that we worked in class yesterday.

2. Suppose that a and b are positive integers. Explain why $\gcd(a^2, b^2)$ cannot be equal to 5.

   Solution. The greatest common divisor of two positive integers equals the product of the prime factors that they have in common (counting repetitions). The prime factorization of a^2 is the square of the prime factorization of a, and similarly for b. Therefore $\gcd(a^2, b^2) = [\gcd(a, b)]^2$. But 5 is not the square of an integer, so 5 cannot be the greatest common divisor of a^2 and b^2.