Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Solve the pair of simultaneous linear congruences

$$\begin{cases} x \equiv 5 \mod 7 \\ x \equiv 10 \mod 13. \end{cases}$$

Solution. The first step is to use the Euclidean algorithm to find a linear combination of the two moduli that is equal to 1. Here is the computation using the matrix method.

$$\begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & 7 \end{pmatrix} \xrightarrow{R1 \to R1 - R2} \begin{pmatrix} 1 & -1 & 6 \\ 0 & 1 & 7 \end{pmatrix} \xrightarrow{R2 \to R2 - R1} \begin{pmatrix} 1 & -1 & 6 \\ -1 & 2 & 1 \end{pmatrix}$$

The deduction is that $(-1) \times 13 + 2 \times 7 = 1$. Thus $(-1) \times 13 \equiv 1 \mod 7$ and $2 \times 7 \equiv 1 \mod 13$. Therefore one value of x that solves the problem is $x = 5 \times (-1) \times 13 + 10 \times 2 \times 7 = 75$.

Remark Other solutions have the form $75 + 91k$ for integers k. The solution can be viewed as the congruence class $[75]_{91}$ in $\mathbb{Z}_{91}$ (where $91 = 7 \times 13$).

2. Which of the values $\phi(25)$ and $\phi(27)$ of Euler's totient function is the greater?

Solution. You know that $\phi(p^k) = p^k - p^{k-1}$ when p is a prime number. Therefore $\phi(25) = \phi(5^2) = 5^2 - 5^1 = 20$, and $\phi(27) = \phi(3^3) = 3^3 - 3^2 = 18$. Thus $\phi(25)$ is greater than $\phi(27)$.