

Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Suppose $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$. Determine the product $\pi\sigma$ of these two permutations in the symmetric group $S(5)$.

Solution. You should be able to write down the answer by inspection, remembering that multiplication of permutations is really function composition, so the action is from right to left:

$$\pi\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix} \quad \text{or} \quad (1\ 4\ 3\ 5)$$

in two alternate notations.

If you are not getting that answer, then look at the following table, which illustrates in steps how the product works:

$$\begin{array}{l} 1 \xrightarrow{\sigma} 2 \xrightarrow{\pi} 4 \\ 2 \xrightarrow{\sigma} 4 \xrightarrow{\pi} 2 \\ 3 \xrightarrow{\sigma} 1 \xrightarrow{\pi} 5 \\ 4 \xrightarrow{\sigma} 3 \xrightarrow{\pi} 3 \\ 5 \xrightarrow{\sigma} 5 \xrightarrow{\pi} 1 \end{array}$$

2. Determine the order of the product $(1\ 2)(1\ 2\ 3)(1\ 2\ 3\ 4)$ of non-disjoint cycles in the symmetric group $S(4)$.

Solution. There is a rule for a product of *disjoint* cycles stating that the order equals the least common multiple of the lengths of the cycles. But the given cycles are not disjoint, so you need to rewrite the product. By inspection, the given product simplifies to $(1\ 3\ 4)$, a three-cycle. Therefore the order equals 3.

If you are not getting that answer, then consult the following table showing the step-by-step computation of the product. In the table, $\pi = (1\ 2)$, $\rho = (1\ 2\ 3)$, and $\sigma = (1\ 2\ 3\ 4)$.

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$$\begin{aligned} 1 &\xrightarrow{\sigma} 2 \xrightarrow{\rho} 3 \xrightarrow{\pi} 3 \\ 2 &\xrightarrow{\sigma} 3 \xrightarrow{\rho} 1 \xrightarrow{\pi} 2 \\ 3 &\xrightarrow{\sigma} 4 \xrightarrow{\rho} 4 \xrightarrow{\pi} 4 \\ 4 &\xrightarrow{\sigma} 1 \xrightarrow{\rho} 2 \xrightarrow{\pi} 1 \end{aligned}$$

Thus the product permutation is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ in two-line notation or $(1\ 3\ 4)$ in cycle notation.