

Applied Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. State the definition of a group.

Solution. A group is a set equipped with an associative binary operation such that there is an identity element and such that every element has an inverse. See page 170 in the textbook.

2. Consider the set of positive integers equipped with the following binary operation: $a * b$ equals the least common multiple of a and b . Does this operation provide a group structure on the positive integers? Explain.

Solution. Since $\text{lcm}(1, n) = n$ for every positive integer n , the element 1 serves as the identity element for this operation. But inverses are lacking: the least common multiple of n and any other number is at least as large as n , so n does not have an inverse when $n > 1$. Therefore the structure described in the problem is not a group.

Remark The operation is associative, as can be seen by thinking about how the least common multiple is computed from prime factorizations (see Corollary 1.3.5 on pages 30–31). Since the existence of inverses is the only property lacking, the problem provides an example of what you learned in class today to call a monoid.