Examination 2

1. Define
   (a) the meaning of a *component* of a topological space (X, τ),
   (b) the meaning of two metrics on a set X being *equivalent* metrics.

2. Give an example of a disconnected metric space (X_1, d_1) and a connected metric space (X_2, d_2) and a continuous mapping $f : X_1 \to X_2$.

3. Suppose X is the two-point space $\{a, b\}$ equipped with the topology consisting of the three sets $\emptyset$, X, and $\{a\}$. Is this topological space path-connected? Explain why or why not.

4. Suppose $f(x) = \cos(x)$ when $x \in \mathbb{R}$. Show that if f is viewed as a mapping from the topological space ($\mathbb{R}$, finite-closed topology) into itself, then f is not continuous.

5. Suppose X is the set of letters of the Greek alphabet. Prove that if d is an arbitrary metric on X, then the topology induced by d must be the discrete topology.

Optional Extra Credit Problem

Given a collection of functions from a set X_1 to a set X_2, and given a topology τ_2 on X_2, there is a natural way to create a topology τ_1 on X_1: namely, the coarsest topology for which all the specified functions are continuous. (A topology τ is coarser than a topology σ if $\tau \subseteq \sigma$, that is, every τ-open set is also σ-open.)

Specifically, suppose that both X_1 and X_2 are $\mathbb{R}$, and τ_2 is the Euclidean topology. Consider the collection of step functions $\{ f_c : c \in \mathbb{R} \}$ of the following form:

$$f_c(x) = \begin{cases} 0, & \text{if } x < c, \\ 1, & \text{if } x \geq c. \end{cases}$$

Determine the coarsest topology τ_1 on X_1 that makes all of these step functions continuous as functions from (X_1, τ_1) to (X_2, τ_2).