New concept: homeomorphism

A function $f\colon (X_1, \tau_1) \to (X_2, \tau_2)$ is a *homeomorphism* if f is a bijection such that $A \in \tau_1 \iff f(A) \in \tau_2$.

Example

The natural numbers $\mathbb{N}$ with the topology $\{\varnothing, \mathbb{N}, \{1\}\}$ and the natural numbers with the topology $\{\varnothing, \mathbb{N}, \{5\}\}$ are homeomorphic spaces.
The real numbers $\mathbb{R}$ and the interval $(-\pi/2, \pi/2)$ with the Euclidean topology are homeomorphic: take f to be the arctangent function.

Properties that can be described in terms of open sets and closed sets are preserved by homeomorphisms: for example, connectedness, interior, closure, neighborhood of a point, basis for the topology, T_0, T_1, saturated set, dense subset, separability, second countable space, to be continued.

Consistency questions for the subspace topology

1. If Y is a subspace of X, and A is a *closed* subset of X, is $A \cap Y$ a relatively closed subset of Y? Yes.
2. Suppose (Y, τ_Y) is a subspace of (X, τ). If $\mathcal{B}$ is a basis for the topology τ, is $\{ B \cap Y : B \in \mathcal{B} \}$ a basis for the subspace topology τ_Y? Yes.

Assignment due next class

Write solutions to the following two consistency questions for the subspace topology.

1. Number 4 in Exercises 4.1 (a subspace of a subspace is a subspace).
2. Suppose (Y, τ_Y) is a subspace of (X, τ). The Cartesian product $Y \times Y$ can be given the product topology corresponding to τ_Y. Alternatively, $Y \times Y$ gets a topology as a subspace of the product space $X \times X$. Are these two topologies on $Y \times Y$ the same?