Local versus global

A topological space (X, τ) is *locally connected* if the topology τ has a basis consisting of connected sets.

Examples

- $(\mathbb{R}^n$, Euclidean topology$)$ is locally connected: open balls are connected and form a basis for the topology.
- $(\mathbb{R}$, discrete topology$)$ is locally connected: singleton sets are connected and form a basis for the topology.
  But this locally connected space is not connected.
- The topologist's sine curve is connected but not locally connected.

More on components

Theorem

If a topological space is locally connected, then its components are necessarily open.

Proof.

Let C be a component and x a point in C. The goal is to show that $x \in \text{Int}(C)$.

By hypothesis, there is *some* connected open set U such that $x \in U$. But C is the union of *all* connected sets containing x. So $U \subseteq C$. Thus C contains a neighborhood of x, as required. □

Local path connectedness

A topological space is *locally path-connected* if the topology has a basis consisting of path-connected sets.

Example

- $(\mathbb{R}^n, \text{Euclidean topology})$ is locally path-connected: open balls are path-connected and form a basis for the topology.
- $(\mathbb{R}, \text{discrete topology})$ is locally path-connected: singleton sets are path-connected and form a basis for the topology. But this locally path-connected space is not path-connected.
- The union of the topologist's sine curve with $\{ (x, 1) : x \in \mathbb{R} \}$ is path-connected but not locally path-connected.

Path-connected and connected are sometimes the same

Theorem

If a topological space is locally path-connected, then components and path-components are the same (and are clopen).

Example

In $\mathbb{R}^n$, connected and path-connected are equivalent for **open** sets (since open sets in $\mathbb{R}^n$ are locally path-connected).

Assignment over Spring Break

Travel safely, remain connected, and find a continuous path back to College Station.