Metric spaces

A *metric*, often denoted d (for “distance”), on a set X is a real-valued function satisfying the following properties.

1. $\forall x \in X$, $\forall y \in X$,
   (a) $d(x, y) = 0 \iff x = y$.
   (b) $d(x, y) \geq 0$.
2. Symmetry: $\forall x \in X$, $\forall y \in X$, $d(x, y) = d(y, x)$.
3. Triangle inequality: $\forall x \in X$, $\forall y \in X$, $\forall z \in X$,
   $d(x, z) \leq d(x, y) + d(y, z)$.

Examples or not?

Which of the following are examples of metric spaces?

1. $X = \mathbb{R}$, $d(x, y) = |x - y|$ yes
2. $X = \mathbb{Q}$, $d(x, y) = |x - y|$ yes (subspace)
3. $X = \mathbb{R}$, $d(x, y) = |x^3 - y^3|$ yes
4. $X = \mathbb{R}$, $d(x, y) = |x^2 - y^2|$ no, because $d(-1, 1) = 0$
5. $X = \mathbb{N}$, $d(n, m) = |n^2 - m^2|$ yes
6. $X = \mathbb{N}$, $d(n, m) = (n - m)^2$ no: $d(1, 3) > d(1, 2) + d(2, 3)$

Every metric space has a natural topology

In a metric space (X, d), an *open ball* with center c and radius r means $\{ x \in X : d(x, c) < r \}$.
Notation: $B_r(c)$.

The collection of open balls is a basis for a topology on X.
Every open set is a union of open balls.

Every metric space is a topological space, but not every topological space is a metric space.

More examples of metric spaces

1. Let $d(x, x) = 0$, and $d(x, y) = 1$ when $x \neq y$. This metric corresponds to the discrete topology.

Assignment due next class

1. Read the first part of section 6.1, through Example 6.1.8.
2. Prove that by using the symmetry property of a metric together with the triangle inequality, it is possible to deduce the property that $d(x, y) \geq 0$ for all x and y.
   (In other words, the requirement that a metric be non-negative is a superfluous axiom.)