Recap

A metric on a set X is a real-valued function satisfying the following properties.

1. $\forall x \in X$, $\forall y \in X$,
   (a) $d(x, y) = 0 \iff x = y$.
   (b) $d(x, y) \geq 0$. [Can be deduced from the other properties.]
2. Symmetry: $\forall x \in X$, $\forall y \in X$, $d(x, y) = d(y, x)$.
3. Triangle inequality: $\forall x \in X$, $\forall y \in X$, $\forall z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

A metric provides a topology, for which a basis consists of all open "balls" $B_r(c) := \{ x \in X : d(x, c) < r \}$.

Some more examples of metric spaces

1. [from last time] Let $d(x, x) = 0$, and $d(x, y) = 1$ when $x \neq y$. This metric corresponds to the discrete topology on the set X.
2. $X = \mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (Euclidean metric).
3. $X = \mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ (taxicab metric).
4. $X =$ the set of continuous real-valued functions defined on the interval $[0, 1]$,
   $d(f, g) = \left(\int_0^1 (f(x) - g(x))^2 \, dx\right)^{1/2}$.
5. $X =$ the set of 2×3 matrices with real entries,
   $d(M_1, M_2) =$ the rank of the matrix $(M_1 - M_2)$
   (the number of linearly independent rows).

Metric spaces are T_2 (Hausdorff) spaces

For if x and y are any two distinct points, then the open balls centered at x and y of radius $\frac{d(x, y)}{3}$ are disjoint.

Discussion of Example 6.1.9: normed vector spaces

A *norm* on a vector space V (with either real or complex scalars) is a real-valued function $\| \cdot \|$ satisfying the following properties.

1. $\forall v \in V$,
   (a) $\|v\| = 0 \iff v$ is the zero vector.
   (b) $\|v\| \geq 0$. [Can be deduced from the other properties.]
2. Homogeneity: $\forall v \in V$ and for every scalar λ, $\|\lambda v\| = |\lambda| \, \|v\|$.
3. Triangle inequality: $\forall v \in V$, $\forall w \in V$, $\|v + w\| \leq \|v\| + \|w\|$.

The model example: $V = \mathbb{R}^2$ or $\mathbb{R}^3$, and $\|v\| =$ the length of the vector v.

A norm induces a metric via $d(v, w) = \|v - w\|$.

Assignment due next class

1. Continue reading section 6.1, through Definition 6.1.22.
2. Write a solution to number 2 in Exercises 6.1.