Compact versus closed

Examples

- In $\mathbb{R}$ with the Euclidean topology, the subset $\mathbb{N}$ is closed but not compact.
- In $\mathbb{R}$ with the finite-closed topology, the subset $\mathbb{N}$ is compact but not closed.

Theorems

- The intersection of a closed set and a compact set is compact.
- A compact subset of a *Hausdorff* space is closed. In particular, a compact subset of a metric space is closed, since metric spaces are examples of Hausdorff spaces.

An important general question about compactness

Given a concrete topological space (X, τ), can the compact subsets of X be characterized via some easily checkable property?

Examples

- For an arbitrary set X with the discrete topology, the compact subsets are the finite subsets.
- For the natural numbers $\mathbb{N}$ with the initial segment topology, the compact subsets are the finite subsets.

Heine–Borel theorem for $\mathbb{R}$: when are subsets compact?

Theorem

A subset of the real numbers
(with the standard Euclidean topology)
is compact if and only if the set is both closed and bounded.

Proof.

$\square$

Assignment due next class

Write a solution to number 1 in Exercises 7.2.