Compactness in metric spaces

Theorem (Part of Exercise 7.2.22)

A **metric** *space is compact if and only if the space is* **sequentially compact**, *that is, every sequence has a convergent subsequence.*

Restatement for subsets:

Theorem

A subset E of a metric space is compact if and only if every sequence of points of E has a subsequence that converges to a point of E.

The proof requires several steps.

A compact metric space is sequentially compact

Proof.

Seeking a contradiction, suppose there is a sequence that has no convergent subsequence.

Then for each point x in the space, there must be some radius $r(x)$ such that the ball $B_{r(x)}(x)$ contains only a finite number of terms of the sequence.

But the space is compact by hypothesis, so a finite number of these balls cover the space, hence contain all infinitely many terms of the sequence. Contradiction. $\square$

A general fact about compactness

Lemma

To check compactness in a topological space, it suffices to consider open covers made up of basis elements.

Proof.

□

Remark. Exercise 7.1.7* is the much harder statement that it suffices to consider subbasis covers.

Sequentially compact metrics spaces are separable

Proof.

□

Sequentially compact metric spaces are second countable

Proof. □

Assignment due next class

For review (not to hand in):

1. Which of the following properties does $\mathbb{R}$ have when considered with the Euclidean topology?
   (a) closed
   (b) connected
   (c) compact
   (d) Hausdorff
   (e) separable
   (f) second-countable
2. Same question for $\mathbb{R}$ with the finite-closed (cofinite) topology.