Principles of Analysis I

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences. Students in section 500 may substitute problems from part C for problems in part A if they wish.

A Section 500: Do both of these problems.

A.1

In the Euclidean plane $\mathbb{R}^2$, the set of points inside a circle is a *disk*. Prove that every set of non-overlapping disks in the plane is at most countable. (Non-overlapping means that no two disks intersect.)

A.2

Give an example of a sequence (a_n) of real numbers such that

$$\inf_n a_n = 1, \quad \liminf_{n\to\infty} a_n = 2, \quad \limsup_{n\to\infty} a_n = 3, \quad \text{and} \quad \sup_n a_n = 4.$$

B Section 500 and Section 200: Do *two* of these problems.

B.1

The producer of the television show "The Biggest Loser" proposes to define a metric d on the set of Texas A&M students as follows: $d(x, y) =$ the maximum weight in pounds of students x and y if x and y are different students, and $d(x, y) = 0$ if x and y are the same student. Does this proposed d satisfy all the properties of a metric? Explain.

B.2

Constance misremembers the definition of continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x as the following statement:

> For every positive ϵ and for every positive δ we have the inequality $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

What well-known set of functions does Constance's property actually characterize? Explain.

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B.3

The irrational number e/π is approximately equal to 0.865255979432. Does the number e/π belong to the Cantor set? Explain how you know.

B.4

Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in the space ℓ_2 of square-summable sequences of real numbers. Thus

$$x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots), \qquad \text{and} \qquad \|x^{(n)}\|_2 = \left(\sum_{k=1}^{\infty} |x_k^{(n)}|^2 \right)^{1/2}.$$

Eleanor conjectures that $x^{(n)} \to 0$ in the normed space ℓ_2 if and only if $x_k^{(n)} \to 0$ in $\mathbb{R}$ for every k. Either prove or disprove Eleanor's conjecture.

C Section 200: Do both of these problems.

C.1

For which values of p does the parallelogram law

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2$$

hold for all elements x and y in the sequence space ℓ_p? Explain.

C.2

Alfie proposes the following "proof" that the real numbers between 0 and 1 form a countable set (a statement that we know to be false):

> The decimals that have exactly one non-zero digit form a countable set; the decimals that have exactly two non-zero digits form a countable set; and so on. The set of all decimals is therefore the union of countably many countable sets, hence is itself a countable set.

Pinpoint the fatal error in Alfie's argument.