Principles of Analysis I

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences.

A Section 500: Do both of these problems.

A.1

In this course, you learned various "c" notions. Some of these concepts are (a) countable, (b) closed, (c) connected, (d) compact, and (e) first category. The Cantor set (viewed as a subset of the real numbers with the standard metric) has which of these properties? Why?
[You may substitute problem C.1 for problem A.1 if you wish.]

A.2

Consider the continuous function $f\colon (0,1) \to \mathbb{R}$ defined by $f(x) = \dfrac{\sin(x)}{x}$. Is this continuous function *uniformly* continuous on the open interval $(0,1)$? Explain why or why not.
[You may substitute problem C.2 for problem A.2 if you wish.]

B Section 500 and Section 200: Do *two* of these problems.

B.1

Suppose $0 \le x \le 1$, and $f_n(x) = \dfrac{x^n}{1+nx}$ when n is a positive integer. Discuss convergence of the sequence (f_n) on the closed interval $[0,1]$. (Does this sequence of functions converge pointwise? uniformly? How do you know?)

B.2

In the metric space $C[0,1]$ of continuous real-valued functions on the closed interval $[0,1]$, let S be the subset consisting of those continuous functions f such that $f(0) = 0$ and $|f(x) - f(y)| \le |x - y|$ for all x and y. Is the set S a *compact* subset of $C[0,1]$? Explain.

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B.3

State the following three theorems: (a) the Bolzano–Weierstrass theorem for real numbers, (b) Hölder's inequality for sequences, and (c) the Weierstrass approximation theorem (any version).

B.4

Suppose (M, d) and (N, ρ) are homeomorphic metric spaces. If M is complete, must N be complete? If M is separable, must N be separable? Explain.

C Section 200: Do both of these problems.

C.1

In this course, you learned various "c" notions. Some of these concepts are (a) countable, (b) closed, (c) connected, (d) compact, and (e) first category. The set of sequences of 0's and 1's, viewed as a subset of the normed space ℓ_∞ of bounded sequences, has which of these properties? Why?

C.2

Let M be the metric space $\mathbb{R}^2 \setminus \{(0,0)\}$ (the "punctured plane") equipped with the standard metric inherited from $\mathbb{R}^2$. Consider the continuous function $f\colon M \to \mathbb{R}$ defined by $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. Is this function *uniformly* continuous on M? Explain why or why not.

D Extra credit (optional) for both Section 500 and Section 200

For bonus points, prove either (a) the Bernstein equivalence theorem about sets of the same cardinality or (b) the Baire category theorem for complete metric spaces.