Principles of Analysis I

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences. Students in section 500 may substitute problems from part C for problems in part A if they wish.

A Section 500: Do both of these problems.

A.1

In the Euclidean plane $\mathbb{R}^2$, the set of points inside a circle is a *disk*. Prove that every set of non-overlapping disks in the plane is at most countable. (Non-overlapping means that no two disks intersect.)

Solution This problem is similar to homework exercise 15 on page 22. Here are two different approaches.

Method 1. We seek an injective function from a given set of disks into the countable set $\mathbb{Q} \times \mathbb{Q}$. The idea is to pick in each disk a point both of whose coordinates are rational numbers. Since the disks are disjoint, no pair of rational numbers is repeated for different disks, so the association disk $\mapsto$ (element of $\mathbb{Q} \times \mathbb{Q}$) is one-to-one.

If you are in doubt that we *can* pick a point in each disk whose coordinates are rational numbers, here are some further details. We can fit inside a given disk some square containing the center of the disk. Suppose this square has the form $[a, b] \times [c, d]$. The rational numbers are dense in $\mathbb{R}$, so there is a rational number q_1 in the interval $[a, b]$ and a rational number q_2 in the interval $[c, d]$. The point whose coordinates are q_1 and q_2 will serve.

We can even give a rule for selecting a definite point with rational coordinates from each disk. Namely, enumerate the elements of $\mathbb{Q} \times \mathbb{Q}$. Select from each disk the first element in the listing of $\mathbb{Q} \times \mathbb{Q}$ that lies in that disk.

Method 2. This method presupposes that we know about the concept of area. For each natural number n, let S_n denote the set of those disks in our given collection that have radius exceeding $1/n$ and that also lie inside the square $[-n, n] \times [-n, n]$. Since each disk in S_n has area at least π/n^2, while the square has area $4n^2$, and the disks do not overlap, the set S_n must be finite. (Certainly S_n contains fewer than $4n^4$ disks.) Now every disk in our collection has a positive radius and so belongs to S_n for sufficiently large n. Hence our set of disks can be realized as the union of countably many finite sets, so it is a countable set.

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A.2

Give an example of a sequence (a_n) of real numbers such that

$$\inf_n a_n = 1, \quad \liminf_{n\to\infty} a_n = 2, \quad \limsup_{n\to\infty} a_n = 3, \quad \text{and} \quad \sup_n a_n = 4.$$

Solution There are many examples. Perhaps the simplest example is the sequence 1, 4, 2, 3, 2, 3, . . . , with the numbers 2 and 3 repeating. A popular correct answer is the following:

$$a_n = \begin{cases} 2 - \frac{1}{n}, & \text{if } n \text{ is odd,} \\ 3 + \frac{2}{n}, & \text{if } n \text{ is even.} \end{cases}$$

If you are leery of piecewise defined sequences, you could use a more elaborate example like the following:

$$a_n = \frac{1}{2}\left[5 + (-1)^n \left(1 + \frac{2^n}{n!}\right)\right].$$

B Section 500 and Section 200: Do *two* of these problems.

B.1

The producer of the television show "The Biggest Loser" proposes to define a metric d on the set of Texas A&M students as follows: $d(x, y) =$ the maximum weight in pounds of students x and y if x and y are different students, and $d(x, y) = 0$ if x and y are the same student. Does this proposed d satisfy all the properties of a metric? Explain.

Solution It is immediate from the indicated definition that $d(x, y)$ is (i) non-negative, (ii) equal to 0 if and only if $x = y$, and (iii) symmetric. What needs to be checked is the triangle inequality:

$$d(x, y) \le d(x, z) + d(z, y).$$

Let's use the ad hoc notation that $\bar{x}$ denotes the weight of student x; thus $d(x, y) = \max\{\bar{x}, \bar{y}\}$ when $x \neq y$. For any d satisfying properties (i), (ii), and (iii), the triangle inequality is a triviality in the special case that $x = y$ or $y = z$ or $z = x$. Therefore we need only consider the case that x, y, and z are all distinct. In that case,

$$\begin{aligned} d(x, y) &= \max\{\bar{x}, \bar{y}\} \leq \bar{x} + \bar{y} \\ &\qquad \text{(the max of positive quantities is less than the sum)} \\ &\leq \max\{\bar{x}, \bar{z}\} + \max\{\bar{y}, \bar{z}\} \\ &\qquad \text{(a number does not exceed its max with another number)} \\ &= d(x, z) + d(y, z). \end{aligned}$$

Thus the triangle inequality holds.

Instead of using the preceding trickery, you could argue in a more pedestrian way as follows. Since both sides of the triangle inequality are symmetric in x and y, there is no loss of generality in assuming that $\bar{x} \leq \bar{y}$. In that case, the left-hand side of the triangle inequality equals $\bar{y}$ (still under the assumption that x, y, and z are all distinct). But $\bar{y} \leq d(y, z)$, so we deduce that $d(x, y) \leq d(y, z)$, and a fortiori $d(x, y) \leq d(x, z) + d(y, z)$.

Another popular method was to consider cases: $\bar{x} \leq \bar{y} \leq \bar{z}$, $\bar{x} \leq \bar{z} \leq \bar{y}$, and $\bar{z} \leq \bar{x} \leq \bar{y}$. At first sight, it appears that there should be six cases, but you can reduce to three cases by symmetry considerations.

Consequently, the proposed d does satisfy all the properties of a metric.

B.2

Constance misremembers the definition of continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point x as the following statement:

> For every positive ϵ and for every positive δ we have the inequality $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

What well-known set of functions does Constance's property actually characterize? Explain.

Solution Constance's property characterizes the constant functions!

It is obvious that every constant function f does satisfy the property, since then $|f(x) - f(y)| = 0 < \epsilon$ for every positive ϵ. Conversely, suppose f is a function satisfying Constance's property. Let y be an arbitrary point. Taking δ equal to $|x-y|+1$ in Constance's property shows that $|f(x)-f(y)|<\epsilon$ for every positive ϵ. The only non-negative real number smaller than *every* positive number is 0, so $|f(x)-f(y)|=0$. Thus $f(y)$ is equal to $f(x)$ whatever y is, so f is a constant function.

B.3

The irrational number e/π is approximately equal to 0.865255979432. Does the number e/π belong to the Cantor set? Explain how you know.

Solution The first step of the construction of the Cantor set removes the points between 1/3 and 2/3. The second step removes the points between 1/9 and 2/9 and also the points between 7/9 and 8/9. Since

$$\tfrac{7}{9}=0.777\ldots<0.865255979432\ldots<0.888\ldots=\tfrac{8}{9},$$

the number e/π is removed at the second stage. Thus the number e/π is not in the Cantor set.

B.4

Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in the space ℓ_2 of square-summable sequences of real numbers. Thus

$$x^{(n)}=(x_1^{(n)},x_2^{(n)},\ldots),\qquad \text{and}\qquad \|x^{(n)}\|_2=\left(\sum_{k=1}^{\infty}|x_k^{(n)}|^2\right)^{1/2}.$$

Eleanor conjectures that $x^{(n)}\to 0$ in the normed space ℓ_2 if and only if $x_k^{(n)}\to 0$ in $\mathbb{R}$ for every k. Either prove or disprove Eleanor's conjecture.

Solution Since $|x_k^{(n)}|\le\|x^{(n)}\|_2$, it is true that $x_k^{(n)}\to 0$ for every k whenever $\|x^{(n)}\|_2\to 0$. The converse, however, is false. For instance, if $x^{(n)}$ is the nth "unit basis vector" $(0,\ldots,0,1,0,\ldots)$ having a 1 in the nth position and 0 elsewhere, then $x_k^{(n)}\to 0$ for each k, yet $\|x^{(n)}\|_2=1\not\to 0$. This example is actually in the textbook at the bottom of page 47.

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C Section 200: Do both of these problems.

C.1

For which values of p does the parallelogram law

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2$$

hold for all elements x and y in the sequence space ℓ_p? Explain.

Solution The parallelogram law holds if and only if $p = 2$.

That the law does hold when $p = 2$ follows from a computation:

$$\begin{aligned}
\|x+y\|_2^2 + \|x-y\|_2^2 &= \sum_{j=1}^{\infty} |x_j + y_j|^2 + \sum_{j=1}^{\infty} |x_j - y_j|^2 \\
&= \sum_{j=1}^{\infty} (x_j^2 + 2x_j y_j + y_j^2) + \sum_{j=1}^{\infty} (x_j^2 - 2x_j y_j + y_j^2) \\
&= 2\sum_{j=1}^{\infty} (x_j^2 + y_j^2) = 2\|x\|_2^2 + 2\|y\|_2^2.
\end{aligned}$$

To see that the law fails when $p \neq 2$ requires a counterexample. Take x and y equal to the first two "unit basis vectors": $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, 0, \ldots)$. Then $\|x+y\|_p^2 = \|x-y\|_p^2 = (1^p + 1^p)^{2/p} = 2^{2/p}$, so the left-hand side of the parallelogram law equals $2 \cdot 2^{2/p}$. On the other hand, the right-hand side of the parallelogram law equals $2 \cdot 1 + 2 \cdot 1 = 2 \cdot 2$. Evidently $2 \cdot 2^{2/p}$ and $2 \cdot 2$ are different when $p \neq 2$.

Remark We know that a norm $\|\cdot\|$ generates a metric via $d(x, y) = \|x-y\|$, but not every metric arises from a corresponding norm. Analogously, an inner product $\langle \cdot, \cdot \rangle$ generates a norm via $\|x\| = \sqrt{\langle x, x \rangle}$, but not every norm arises from a corresponding inner product. It turns out that a norm does arise from some inner product precisely when the norm satisfies the parallelogram law.

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C.2

Alfie proposes the following "proof" that the real numbers between 0 and 1 form a countable set (a statement that we know to be false):

> The decimals that have exactly one non-zero digit form a countable set; the decimals that have exactly two non-zero digits form a countable set; and so on. The set of all decimals is therefore the union of countably many countable sets, hence is itself a countable set.

Pinpoint the fatal error in Alfie's argument.

Solution Alfie's fatal oversight is that there are lots of real numbers having infinitely many non-zero decimal digits, such as $0.101010\ldots$ and $\pi - 3$. Alfie has counted only the decimals having a finite number of non-zero digits, and these constitute a (proper) subset of the rational numbers.