Principles of Analysis I

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences. Students in section 500 may substitute problems from part C for problems in part A if they wish.

A Section 500: Do both of these problems.

A.1

Give an example of a metric space that is neither connected, nor totally bounded, nor complete. Say why your example has the required properties.

Solution. One example is the rational numbers $\mathbb{Q}$ with the standard metric. This space is disconnected because it is covered by the two nonempty, disjoint, open subsets $\{x \in \mathbb{Q} : x < \sqrt{2}\,\}$ and $\{x \in \mathbb{Q} : x > \sqrt{2}\,\}$. The space is not bounded, so it is certainly not totally bounded. The space is not complete, because a sequence of rational numbers converging in $\mathbb{R}$ to $\sqrt{2}$ is a Cauchy sequence that does not converge to a point of $\mathbb{Q}$.

A.2

Consider the space $C[0,1]$ of continuous functions on the closed interval $[0,1]$ provided with the standard metric: $d(f,g) = \max_{0\le x\le 1} |f(x) - g(x)|$. Let $L\colon C[0,1] \to \mathbb{R}$ be the function defined via $L(f) = \int_0^1 f(x)\sin(x)\,dx$. Prove that L is a continuous function.

Remark This problem is an instance of the general fact that the Fourier coefficients of a function depend continuously on the function.

Solution. Observe that

$$\begin{aligned} |L(f) - L(g)| &= \left| \int_0^1 [f(x) - g(x)]\sin(x)\,dx \right| \le \int_0^1 |f(x) - g(x)|\,dx \\ &\le \int_0^1 d(f,g)\,dx = d(f,g). \end{aligned}$$

Consequently, we can be sure that $|L(f) - L(g)| < \varepsilon$ whenever $d(f,g) < \varepsilon$. In other words, the definition of continuity is satisfied with δ chosen to be equal to ε.

Remark. In the terminology that we recently learned, the function L is uniformly continuous, since the choice of δ depends only on ε and not on f. In fact, the function L satisfies a Lipschitz condition in the sense of Exercise 19 on page 66.

B Section 500 and Section 200: Do *two* of these problems.

B.1

(a) Is it true in every metric space that every closed set is equal to the closure of its interior? Give either a proof or a counterexample.

(b) Is it true in every metric space that every open set is equal to the interior of its closure? Give either a proof or a counterexample.

Solution.

(a) If a nonempty closed set has empty interior, then the set is not equal to the closure of its interior! Some examples of such sets in $\mathbb{R}$ (with the standard metric) are finite sets of points, the set of integers, and the Cantor set.

(b) An open set is certainly contained in the interior of its closure, but the interior of the closure can be strictly larger than the original open set. The complements of the examples in part (a) are examples of this phenomenon. Each of these complements is an open subset of $\mathbb{R}$ whose closure is all of $\mathbb{R}$, so the interior of the closure is equal to $\mathbb{R}$.

B.2

Suppose (M, d) is a complete metric space containing at least two points, and suppose there is a point x_0 in M such that $(M \setminus \{x_0\}, d)$ is a complete metric space too. Prove that M is disconnected.

Solution. I claim that there exists a positive radius r such that $\{ x \in M : d(x, x_0) < r \} = \{x_0\}$. Indeed, if no such r exists, then there is a sequence (x_n) in M such that each x_n is different from x_0 and $d(x_n, x_0) \to 0$. Then (x_n) is a Cauchy sequence in $M \setminus \{x_0\}$ that does not converge to a point of $M \setminus \{x_0\}$, contradicting the completeness of $M \setminus \{x_0\}$.

Thus the singleton $\{x_0\}$ is an open subset of M (in fact, it is an open ball). On the other hand, singletons are closed sets in any metric space. Accordingly, the singleton set $\{x_0\}$ is a nontrivial subset of M that is simultaneously open and closed. Therefore the metric space M is disconnected (by Theorem 6.1 on page 79).

B.3

Consider the following two subsets of the real numbers $\mathbb{R}$ equipped with the standard metric: $\mathbb{N}$ is the set of natural numbers 1, 2, 3, …; and S is the set of reciprocals of natural numbers 1, 1/2, 1/3, …. Show that S is totally bounded, $\mathbb{N}$ is *not* totally bounded, and $\mathbb{N}$ is homeomorphic to S.

Remark This example is a special instance of a general property: namely, a metric space is separable if and only if it is homeomorphic to a totally bounded space.

Solution. To see that S is totally bounded, fix a positive ε. Choose an integer N larger than $1/\varepsilon$. The ball $B_\varepsilon(1/N)$ contains every point $1/n$ for which $n \geq N$. Then S is covered by the finite number of balls $B_\varepsilon(1)$, $B_\varepsilon(1/2)$, …, $B_\varepsilon(1/N)$. Since ε is arbitrary, the set S is totally bounded.

The set $\mathbb{N}$ is not bounded and therefore is not totally bounded. Indeed, a ball of radius $1/3$ in $\mathbb{R}$ contains at most one integer, so no finite number of balls of radius $1/3$ can cover $\mathbb{N}$.

Since the points of $\mathbb{N}$ all are isolated, each point is an open subset of $\mathbb{N}$, so every subset of $\mathbb{N}$ is open relative to $\mathbb{N}$. The set S has the same property for the same reason. Hence any bijection between $\mathbb{N}$ and S is a homeomorphism, for the inverse image of every open set is open, and the image of every open set is open. Consequently, the obvious bijection $n \mapsto 1/n$ is a homeomorphism.

Principles of Analysis I

B.4

In the sequence space ℓ_2 [the space of sequences $x = (x_1, x_2, \ldots)$ with norm $\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$], let S denote the set of absolutely summable sequences [that is, sequences $(x_1, x_2, \ldots)$ for which the series $\sum_{n=1}^{\infty} |x_n|$ converges]. Prove that S is dense in ℓ_2 [that is, the closure of S equals the whole space ℓ_2].

Solution. Fix a point x in ℓ_2, and fix a positive ε. We need to show that there exists an element y in S such that $\|x - y\|_2 < \varepsilon$.

By hypothesis, the series $\sum_{n=1}^{\infty} |x_n|^2$ converges. Therefore there is some N such that $\sum_{n=N}^{\infty} |x_n|^2 < \varepsilon^2$. Define y by setting $y_n = x_n$ when $n < N$, and $y_n = 0$ when $n \geq N$. Obviously $y \in S$, since y has only finitely many nonzero components. Now $\|x - y\|_2^2 = \sum_{n=N}^{\infty} |x_n - 0|^2 < \varepsilon^2$, so $\|x - y\|_2 < \varepsilon$.

C Section 200: Do both of these problems.

C.1

Suppose $f\colon (M, d) \to (N, \rho)$ is a function between metric spaces with the property that for every convergent sequence (x_n) in M, the image sequence $(f(x_n))$ has a convergent *subsequence*. Must f be continuous? Supply a proof or a counterexample, as appropriate.

Solution. Here is a counterexample. Take M and N both equal to $\mathbb{R}$ with the standard metric. Define $f(x)$ to be 0 when x is a rational number, and set $f(x)$ equal to 1 when x is an irrational number. Evidently f is discontinuous. Whatever the sequence (x_n) is (whether convergent or not), the sequence of image values $(f(x_n))$ is a sequence of 0's and 1's, so there must be a convergent subsequence (since either 0 or 1 must appear infinitely often in the sequence). The same argument applies to any function whose range consists of a finite number of points.

C.2

Connie conjectures that the following statement holds in every complete metric space: If (F_n) is a decreasing sequence of nonempty nested sets (that is,

$F_1 \supset F_2 \supset F_3 \supset \cdots$), if the set F_n is both closed and connected for every n, and if the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty, then the intersection $\bigcap_{n=1}^{\infty} F_n$ must be connected. Either prove Connie's conjecture or give a counterexample, as appropriate.

Solution. Here is a counterexample, based on the idea that a jacket holds together when unzipped any finite amount, but falls apart when unzipped all the way.

In $\mathbb{R}^2$, let E_n denote the open half strip $\{\,(x,y) : |x| < 1 \text{ and } y < n\,\}$. The open sets E_n form a nested increasing sequence. Let F_n denote the complementary set $\mathbb{R}^2 \setminus E_n$. The closed sets F_n form a nested decreasing sequence. Evidently each set F_n is connected (for any two points of F_n can be joined by a polygonal path within F_n). The intersection $\bigcap_{n=1}^{\infty} F_n$ consists of the two separated half planes $\{\,(x,y) : x \le -1\,\}$ and $\{\,(x,y) : x \ge 1\,\}$. Thus the intersection of all the sets F_n is nonempty and disconnected.