Principles of Analysis II

In this assignment, you will construct a periodic, continuous function whose Fourier series has points of divergence. The existence of such a function was an open question for half a century after Fourier's work. Paul du Bois-Reymond constructed the first example in the 1870s.

The idea is to begin by creating a trigonometric polynomial of high degree such that a certain partial sum has large size at the origin even though the whole sum is bounded by 1. Forming a convergent series of such trigonometric polynomials of progressively larger degrees produces the final example. The method is an instance of "condensation of singularities" (a term coined by Hermann Hankel in 1870); you saw a different implementation of this technique in Assignment 6.

Recall the following notation from the textbook. If $f \in L_1[-\pi, \pi]$, then $s_n(f)$ denotes the sum of the terms of the Fourier series of f up to degree n. This partial sum of the Fourier series can be written as the convolution of f with the Dirichlet kernel D_n:

$$s_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x - t)f(t)\, dt.$$

The symbol $\sigma_n(f)$ denotes the Cesàro mean $[s_0(f) + \cdots + s_{n-1}(f)]/n$. This trigonometric polynomial can be written as the convolution of f with the Fejér kernel K_n:

$$\sigma_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(x - t)f(t)\, dt.$$

1. Let f be an integrable function such that $|f(x)| \leq 1$ for all x in the interval $[-\pi, \pi]$. Show that $|\sigma_n(f)(x)| \leq 1$ for every positive integer n and for every real number x.

2. Let f be an integrable function such that $|f(x)| \leq 1$ for all x in the interval $[-\pi, \pi]$. Let g_n denote $\sigma_{n^2}(f)$, which is a trigonometric polynomial of degree (at most) $n^2 - 1$. Show that $|s_n(g_n)(x) - s_n(f)(x)| \leq 4$ for every positive integer n and for every real number x.

Recall from the proof of Lemma 15.2(e) that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt > \frac{4\log(n + 1)}{\pi^2}. \tag{†}$$

Principles of Analysis II

The textbook states the inequality with $\log n$ instead of $\log(n + 1)$, but the stronger inequality (†) follows from the same argument: in the last line of the proof on page 252, observe that comparing a sum of areas of boxes with the area under the graph of the function $1/x$ shows that $\sum_{k=1}^{n}(1/k) > \int_{1}^{n+1}(1/x)\,dx = \log(n + 1)$, a slight improvement over the author's $\log n$. (This improved inequality holds for every nonnegative integer n, while the author's inequality has an undefined term when $n = 0$.)

3. Let $f_n(x)$ denote $|D_n(x)|/D_n(x)$ when $D_n(x) \neq 0$, and let $f_n(x)$ be 0 when $D_n(x) = 0$. In other words, $f_n(x)$ is the "sign function" of $D_n(x)$. Thus $D_n(x)f_n(x) = |D_n(x)|$ for all x. Let h_n denote the trigonometric polynomial $\sigma_{n^2}(f_n)$. Show that

$$s_n(h_n)(0) > \frac{4\log(n + 1)}{\pi^2} - 4$$

   for every positive integer n.

4. For each positive integer k, let n_k denote the integer 2^{3^k}. (The form of n_k will be important in part 5: in particular, $n_{k+1} = n_k^3$.) Using the functions h_n from part 3, define a function h as follows:

$$h(x) = \sum_{k=1}^{\infty} 2^{-k} h_{n_k}(n_k x).$$

   Show that h is a continuous function on $\mathbb{R}$, periodic with period 2π.

5. Show that $s_{n_j^2}(h)(0) \to \infty$ as $j \to \infty$, and deduce that the Fourier series of the continuous function h diverges at 0. [Hint: split the sum defining h into three pieces according as $k < j$, $k = j$, or $k > j$.]