Principles of Analysis II

Instructions Solve *four* of the following six problems. Please write your solutions on your own paper. Explain your reasoning in complete sentences.

1. Suppose that A and B are two Borel subsets of the real numbers $\mathbb{R}$. Prove that Lebesgue measure m satisfies the following property:

$$m(A \cap B) + m(A \cup B) = m(A) + m(B).$$

2. Suppose that (E_n) is a sequence of measurable subsets of $\mathbb{R}$. Prove that

$$m\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k\right) \le \liminf_{n\to\infty} m(E_n).$$

3. Prove that every increasing function $f\colon \mathbb{R} \to \mathbb{R}$ is measurable.

   [Notice that f need not be continuous.]

4. Give an example of a sequence (f_n) of measurable functions (from $\mathbb{R}$ into $\mathbb{R}$) converging pointwise to a limit function f but not converging almost uniformly.

   [Such an example shows that the conclusion of Egorov's theorem fails on unbounded intervals. Recall that $f_n \to f$ almost uniformly if for every positive ε there exists a measurable set E of measure less than ε such that $f_n \to f$ uniformly on the complement of E.]

5. Give an example of a sequence (φ_n) of nonnegative simple functions such that $\int \varphi_n \le 1$ for every n, but $\int \varphi_n^2 \to \infty$ as $n \to \infty$.

6. Apply an appropriate convergence theorem for integrals to compute

$$\lim_{n\to\infty} \int_0^1 \frac{n\cos(x)}{1+n^2x^2}\, dx.$$