Principles of Analysis II

Instructions Solve *four* of the following six problems. Please write your solutions on your own paper. Explain your reasoning in complete sentences.

1. Let f be a function of bounded variation on the interval $[0, 1]$. Suppose there is a positive number δ such that $|f(x)| \geq \delta$ for every x (in other words, the function f is bounded away from 0). Show that the reciprocal function $1/f$ is a function of bounded variation.

2. Give a concrete example of a uniformly convergent sequence (f_n) of functions of bounded variation on the interval $[0, 1]$ such that the limit function does not have bounded variation.

3. If α is a nondecreasing function on the closed interval $[-\pi, \pi]$, is it necessarily true that $\lim_{n\to\infty} \int_{-\pi}^{\pi} \cos(nx)\, d\alpha(x) = 0$? (In other words, does the Riemann–Lebesgue lemma carry over to the setting of the Stieltjes integral?) Give either a proof or a counterexample.

4. Let f be a bounded function that is Riemann–Stieltjes integrable with respect to the increasing function α on the interval $[0, 1]$. Prove that f is Riemann–Stieltjes integrable with respect to α^2 on the same interval. In other words, if $\int_0^1 f\, d\alpha$ exists, then so does $\int_0^1 f\, d(\alpha^2)$.

5. Determine the Fourier series of the odd function on the interval $[-\pi, \pi]$ that is equal to 1 on the interval $(0, \pi)$, and use the result to compute the value of the numerical series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

6. Suppose $f \in L_2[-\pi, \pi]$. Then $s_n(f)$, the nth partial sum of the Fourier series of f, has the property that $\lim_{n\to\infty} \|s_n(f) - f\|_2 = 0$ (according to the Riesz–Fischer theorem). Use this result to prove that the Cesàro sum $\sigma_n(f)$, which is the average $[s_0(f) + \cdots + s_{n-1}(f)]/n$, has the corresponding property that $\lim_{n\to\infty} \|\sigma_n(f) - f\|_2 = 0$.

Bonus problem For extra credit, prove either the Riesz representation theorem characterizing the dual space of $C[0, 1]$ or Jordan's decomposition theorem for functions of bounded variation.