

Exercise on uniform convergence

The goal of this exercise is to refresh and to consolidate your understanding of uniform convergence of sequences of functions.

Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions defined on a set E is said to converge *pointwise* to a function f if for every point z in E , the sequence $\{f_n(z)\}_{n=1}^{\infty}$ of complex numbers converges to the number $f(z)$; in other words, for every positive ϵ there exists an integer N (which may depend both on ϵ and on z) such that $|f_n(z) - f(z)| < \epsilon$ when $n > N$. The sequence converges *uniformly* if the integer N in the preceding definition can be chosen to be independent of the point z .

1. Let $f_n(z) = z^n$. Determine the largest open subset of the complex plane \mathbb{C} on which the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise.
2. Show that the sequence $\{z^n\}_{n=1}^{\infty}$ converges uniformly on the open disc $\{z \in \mathbb{C} : |z| < 1/2\}$.
3. Show that there is no largest open set on which the sequence $\{z^n\}_{n=1}^{\infty}$ converges uniformly.

In the theory of functions of a complex variable, one is interested ordinarily in convergence of functions on *open* sets, since open sets are where holomorphic functions live. As the preceding example indicates, there is generally no largest open set on which uniform convergence takes place. Therefore, the following definition is useful: a sequence of functions $\{f_n\}_{n=1}^{\infty}$ is said to converge *locally uniformly* (or *normally*) on an open set if for every point w in the set, the sequence converges uniformly on some disc centered at w .

4. A sequence $\{f_n\}_{n=1}^{\infty}$ of functions converges locally uniformly on an open set G if and only if it converges uniformly on each compact subset of G .

If a sequence of holomorphic functions converges locally uniformly on an open set, then the limit function is holomorphic. This statement follows from Morera's theorem (for instance), because a closed curve γ is a compact set, and $\lim_{n \rightarrow \infty} \oint_{\gamma} f_n(z) dz = \oint_{\gamma} \lim_{n \rightarrow \infty} f_n(z) dz$ for any sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ that converges uniformly on γ .

There is no analogous theorem about differentiability of the limit of a sequence of real-differentiable functions. For example, the partial sums of the Fourier series for $|x|$ on the interval $(-\pi, \pi)$ are C^{∞} (indeed, real-analytic) functions that converge uniformly to the non-differentiable function $|x|$.