Instructions Do any *five* of the following seven problems.

1. Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_{|z|=1}\frac{e^z}{z^{617}}\,dz$, where the integration path is the unit circle oriented in the standard counterclockwise direction.

Solution. By Cauchy's integral formula, the integral represents

$$\frac{1}{616!}\left(\frac{d}{dz}\right)^{616} e^z\bigg|_{z=0}, \qquad \text{or} \qquad \frac{1}{616!}.$$

2. Let f be an entire function such that $\displaystyle\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$ is bounded as a function of the radius r. Prove that f is a constant function.
[This variation on Liouville's theorem is Exercise 7.9 in the textbook.]

Solution. Fix an arbitrary point z, and take R greater than $2|z|$. Then by Cauchy's integral formula,

$$f(z)-f(0)=\frac{1}{2\pi i}\int_{|w|=R}\left(\frac{f(w)}{w-z}-\frac{f(w)}{w}\right)dw=\frac{1}{2\pi i}\int_{|w|=R}\frac{zf(w)\,dw}{w(w-z)}.$$

Since $|w-z|\geq R/2$ when w is on the integration path, it follows that

$$|f(z)-f(0)|\leq\frac{|z|}{\pi R}\int_0^{2\pi}|f(Re^{i\theta})|\,d\theta.$$

Keeping z fixed, let $R\to\infty$ to deduce that $f(z)-f(0)=0$. Since the point z is arbitrary, the function f is identically equal to $f(0)$.

3. One version of the *Stone-Weierstrass theorem* says that if a subset of the continuous, complex-valued functions on a compact set K in the plane contains the constant functions, separates points, and is closed under the formation of linear combinations, products, and complex conjugates, then the subset of functions is dense (with respect to the supremum norm) in the space of all continuous, complex-valued functions on K. In this problem you will show that the hypothesis of closure under conjugation is an essential hypothesis.

Prove that there does *not* exist a sequence of polynomials $\{p_n(z)\}$ such that $p_n(z) \to \overline{z}$ uniformly on the unit circle $\{\, z \in \mathbb{C} : |z| = 1 \,\}$.
[Hint: observe that on the unit circle, $\overline{z}$ is equal to $1/z$.]

Solution. By Cauchy's theorem, $\int_{|z|=1} p_n(z)\,dz = 0$ (because the polynomial p_n is analytic on and inside the integration curve). On the other hand, $\int_{|z|=1} \overline{z}\,dz = \int_{|z|=1} \frac{1}{z}\,dz = 2\pi i$. Therefore

$$\lim_{n\to\infty} \int_{|z|=1} p_n(z)\,dz \neq \int_{|z|=1} \overline{z}\,dz.$$

Consequently, the functions $p_n(z)$ cannot be converging uniformly to $\overline{z}$ on the unit circle (because uniform convergence would imply that the limit of the integrals equals the integral of the limit).

4. This question asks whether the coincidence principle for analytic functions remains valid "at infinity".

Suppose that f and g are entire functions such that $f(n) = g(n)$ for every integer n. Does it follow that $f(z)$ is equal to $g(z)$ for every complex number z? Supply either a proof or a counterexample, as appropriate.

Solution. If $f(z) = \sin(\pi z)$ and $g(z) = z\sin(\pi z)$, then $f(n) = 0 = g(n)$ for every integer n, but f and g are not the same function. Thus the coincidence principle does not hold at infinity.

By adding a growth hypothesis, however, one can obtain a positive result. *Carlson's theorem* says that if f is an entire function such that $|f(z)| \le Ae^{\lambda|z|}$ for all z, where A and λ are positive constants such that $\lambda < \pi$, and if $f(n) = 0$ for every integer n, then f is identically equal to 0.

5. Suppose that f is analytic on the punctured unit disk $\{\, z \in \mathbb{C} : 0 < |z| < 1 \,\}$, and $f(1/n) = (-1)^n$ for $n = 2, 3, \ldots$. Show that the isolated singularity of f at the origin is an essential singularity.

Solution. Since $\{\frac{1}{2n}\}$ and $\{\frac{1}{2n+1}\}$ are two different sequences converging to 0 along which f has different limits, the singularity of f is not removable. Since $|f|$ does not tend to infinity along the sequence $\{\frac{1}{n}\}$, the singularity is not a pole. Therefore the singularity is an essential singularity, since the other possibilities are excluded.

Incidentally, there do exist functions with the indicated property: for instance, $\cos(\pi/z)$ or $\cos^3(\pi/z)$.

6. The Gamma function $\Gamma(z)$ has a simple pole when $z = -1$. Determine the residue at this pole.

Solution. According to the functional equation, $\Gamma(z+1) = z\Gamma(z)$. Therefore the residue at the simple pole at -1 equals

$$\lim_{z\to -1}(z+1)\Gamma(z) = \lim_{z\to -1}\frac{z+1}{z}\Gamma(z+1) = \lim_{z\to -1}\frac{\Gamma(z+2)}{z} = \frac{\Gamma(1)}{-1} = -1.$$

7. Suppose $f(z)$ is the rational function $\dfrac{z^2+1}{z^2-1}$, and C is a simple closed curve that passes through neither of the points 1 and -1. The value of the integral $\int_C f(z)\,dz$ depends on the curve C. Determine all the possible values of this integral.

Solution. Either the curve encloses no poles, or one pole, or both poles; and the curve may be oriented either clockwise or counterclockwise. The residues are 1 at the simple pole where $z = 1$ and -1 at the simple pole where $z = -1$. Consequently, the value of the integral is either 0 (if the curve circles neither pole or both poles) or $\pm 2\pi i$ (if the curve circles one pole but not the other).

A simple curve does not intersect itself, so the curve cannot circle one pole in the clockwise direction and the other pole in the counterclockwise direction. If the curve is not required to be simple, then the integral can equal $2\pi i n$ for any integer n.